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NOTES
ON
STRUCTURAL MECHANICS

FOR THE USE OF STUDENTS
OF THE
LOWELL INSTITUTE
SCHOOL FOR INDUSTRIAL FOREMEN
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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To Capt. Elliott Snow with the Compliments of the Lowell Institute School.

Charles F. Park, Director

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PREFACE

These notes have been prepared for the use of students of the Lowell Institute School for Industrial Foremen, a Free Evening School for young men established by the Trustee of the Lowell Institute, under the auspices of the Massachusetts Institute of Technology.

No attempt has been made to present, in detail, all of the theories involved. It is believed, however, that enough of the common formulae have been derived in placing the fundamentals of Structural Mechanics before the first-year students of the Buildings Course to enable them to follow understandingly the work of the second year in Structural Design.

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NOTES ON STRUCTURAL MECHANICS

PART I

FORCE SYSTEMS

1. Structural Mechanics. The subject of Structural Mechanics deals with the systems forces necessary for the support of any structure and the stresses in its various members due to the loads. The actual proportioning of the structure and its members, the design of proper connections, etc., comes under the head of Structural Design.

The forces acting in connection with a structure may be divided into two groups; the Outer or external forces and the Inner or internal forces. The loads and supports comprise the first group and the stresses in the members and their connections, the second.

If a structure is to be of any service, the forces acting externally to it must form a balanced system. The forces acting externally to any part of the structure must be in equilibrium also.

2. Systems of Forces. The forces acting on a body may be divided into two groups: Those whose lines of action are in the same plane and those whose lines of action are in different planes. Each of these groups may be sub-divided into three divisions: Forces whose lines of action pass through a common point, Forces whose lines of action are parallel, and Forces whose lines of action are neither parallel nor pass through the same point. The last case is general and covers the first two.

There are three important considerations for each system of forces: —

1. Conditions of equilibrium.
2. Number of unknown forces that may be determined when the system is in equilibrium.
3. Determination of the Resultant of the system when it is not in equilibrium.

3. Forces whose Lines of Action are in the Same Plane and Pass Through the Same Point. There are three methods by which problems coming under this case may be solved. Stress Polygon method, method of Resolution, and method of Moments. Any problem can usually be solved more easily by one of these methods than by either of the others.

Stress Polygon Method. A force may be represented by a line (Fig. 1). The line *ab* represents a force whose line of action is along *ab*. The magnitude of this force is represented by the length of (*ab*) which may be drawn to any suitable scale. The arrow indicates the direction in which the force acts.

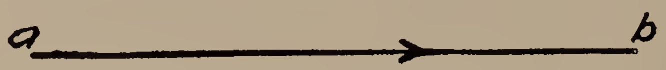


Fig. 1.

Let F_1, F_2, and F_3 (Fig. 2) be three forces whose lines of action are in the same plane and pass through the same point O. Draw a line parallel to and proportional to any one of the three forces, say f_2 (Figs. 3 and 4). Use small letters on the stress polygons. The arrow should indicate the direction in which the force acts with reference to point O. From either end of this line, draw a line parallel and proportional to either of the other forces. Care must be taken to see that the arrows which indicate the direction of the forces with respect to point O follow each other in order, thus (→ → →), and do not reverse their direction thus (→ ← →).

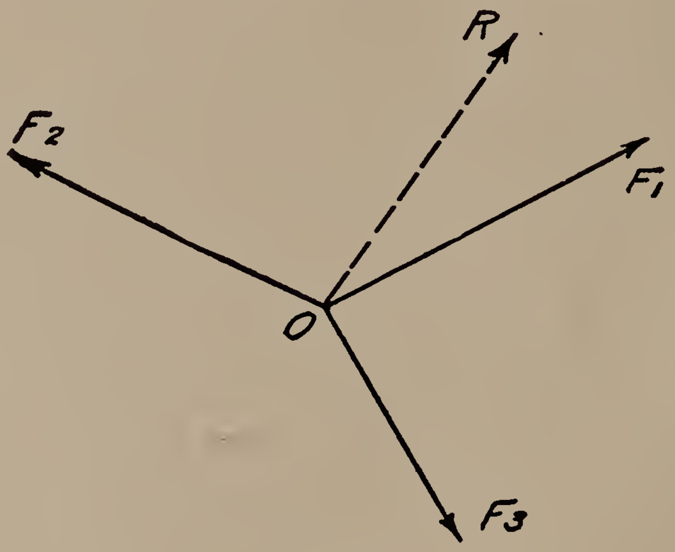


Fig. 2.

The dotted line *ab* will represent the resultant of the three forces F_1, F_2 and F_3. The length of this line will represent the magnitude of the resultant to the same scale as that to which the other forces were drawn. The direction of the resultant, with respect to point O is shown by an arrow which is opposite in direction to the arrows on the lines representing the other forces. The resultant is shown in Fig. 2 by the dotted line OR, which is parallel to the line ab in Figs. 3 and 4. Figs. 3 and 4 are stress polygons for the forces F_1, F_2 and F_3. A number of other stress polygons for these three forces can be constructed and each will give the same resultant.

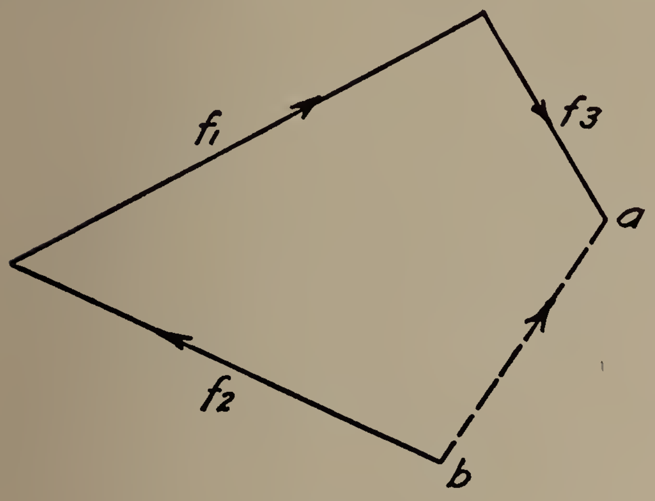


FIG. 3.

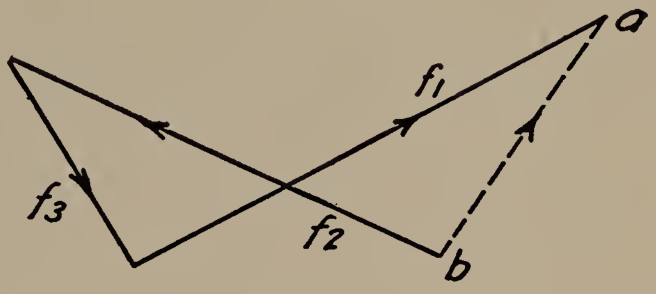


FIG. 4.

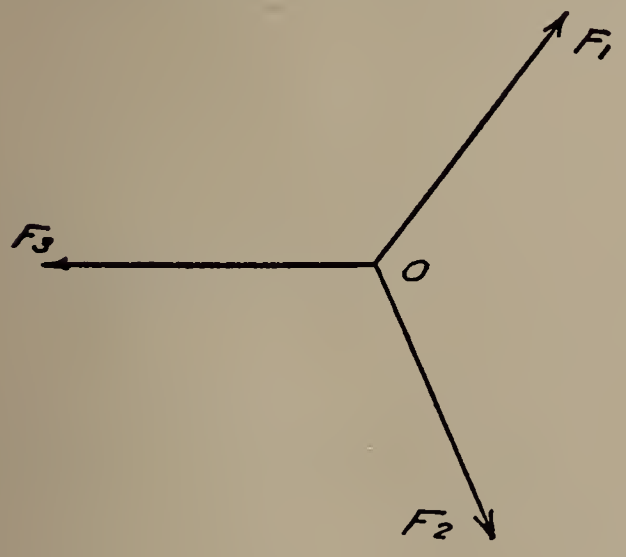


FIG. 5*a*.

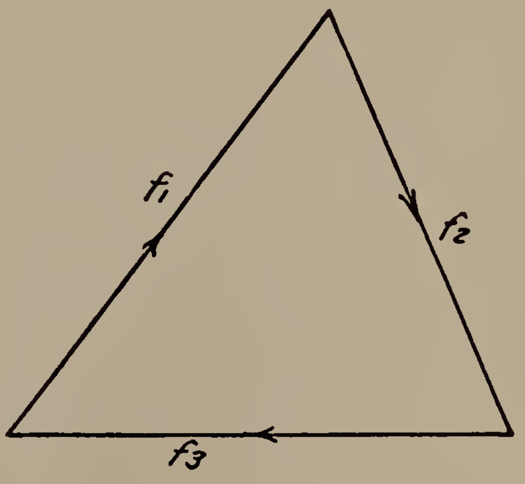


FIG. 5*b*.

Conditions for Equilibrium. If the stress polygon had closed, there would have been no resultant and the forces would have formed a balanced system. (Figs. 5*a* and 5*b*.)

Lines f_1, f_2 and f_3 of the stress polygon in Fig. 5*b* are drawn parallel and proportional to the forces F_1, F_2 and F_3 of Fig. 5*a*.

Determination of Unknown Forces. When forces acting in the same plane and at the same point are in equilibrium and certain of the forces are unknown, the method of the stress polygon may be used for the determination of the unknown forces as follows:

(*a*) Nothing Known about the Unknown Forces. In this case only one unknown force can be determined. This force is equal and opposite to the resultant of the known forces. If more than one unknown force of this character exists, the case becomes what is known as statically indeterminate.

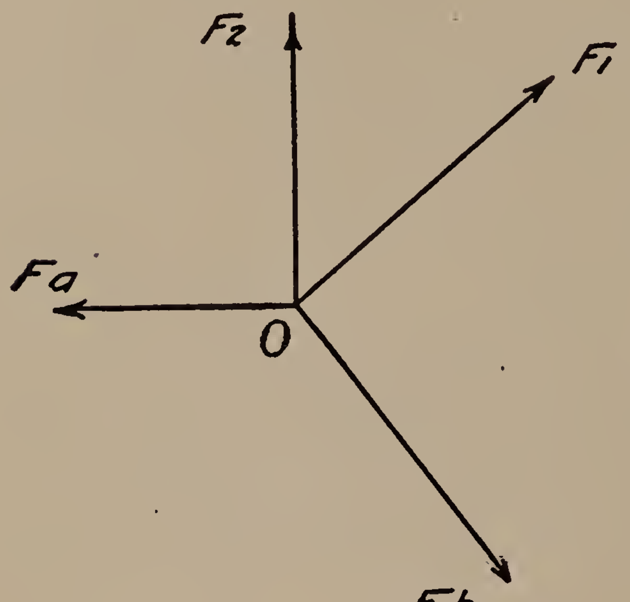


FIG. 6*a*.

(*b*) Lines of Action of the Unknown Forces Determined but Magnitude and Direction Unknown. In this case two unknown forces can be determined. Construct a stress polygon for the known forces, being sure that the arrows follow each other. From the ends of the lines forming this incomplete polygon, draw lines parallel to the unknown forces. (Figs. 6*a* and 6*b*.)

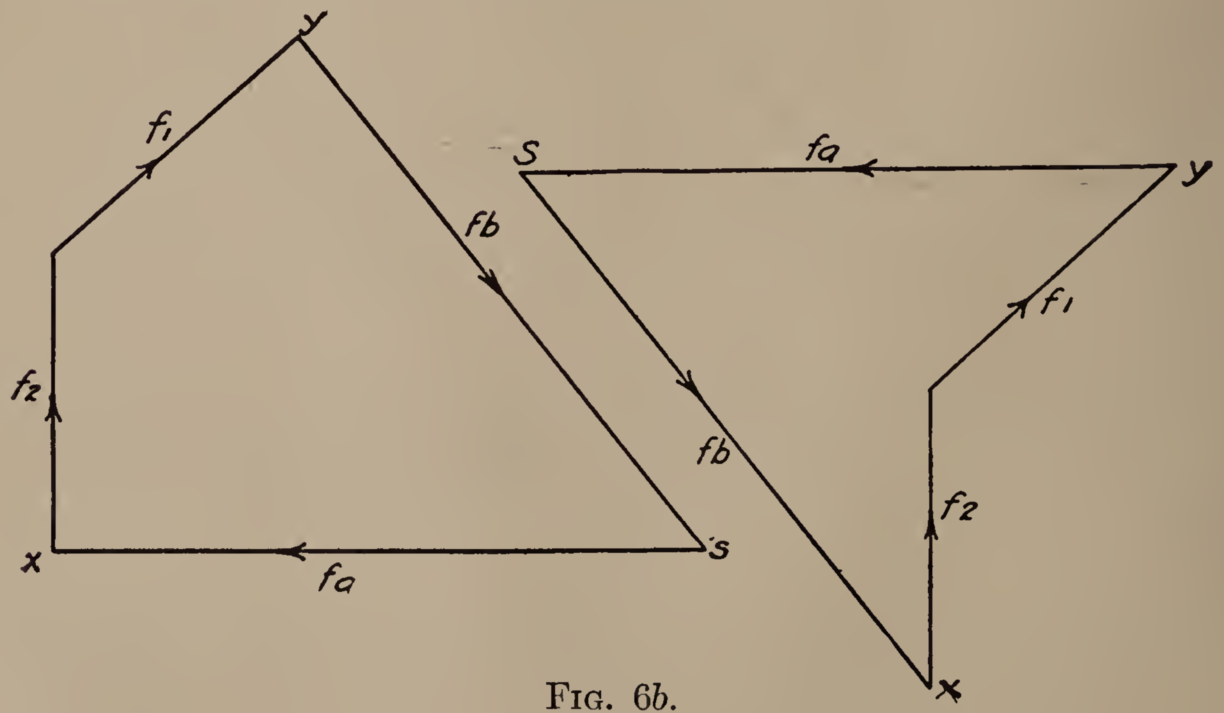


FIG. 6*b*.

f_1 and f_2 are parallel and proportional to tne known forces F_1 and F_2. x and y are the ends of the lines forming the incomplete polygon. f_a and f_b are drawn parallel to the lines of action of the unknown forces F_a and F_b. The intersection of these lines at s gives the magnitude of these unknown forces. The arrows on a closed stress polygon must follow each other in order, therefore, the directions of the unknown forces with respect to O are determined by the arrows as placed upon f_a and f_b as shown in Fig. 6*b*, dotted in Fig. 6*a*.

If there are more than two unknown forces, the case becomes statically indeterminate.

Problems involving the construction of stress polygons may be solved graphically or analytically.

Method of Resolution. If the three forces, F_1, F_2 and F_3 (Fig. 7), are resolved into components along any two coördinate axes in their plane and intersecting at O, the components of their resultant along each of these axes will be the algebraic sum of the components of the forces along that axis.

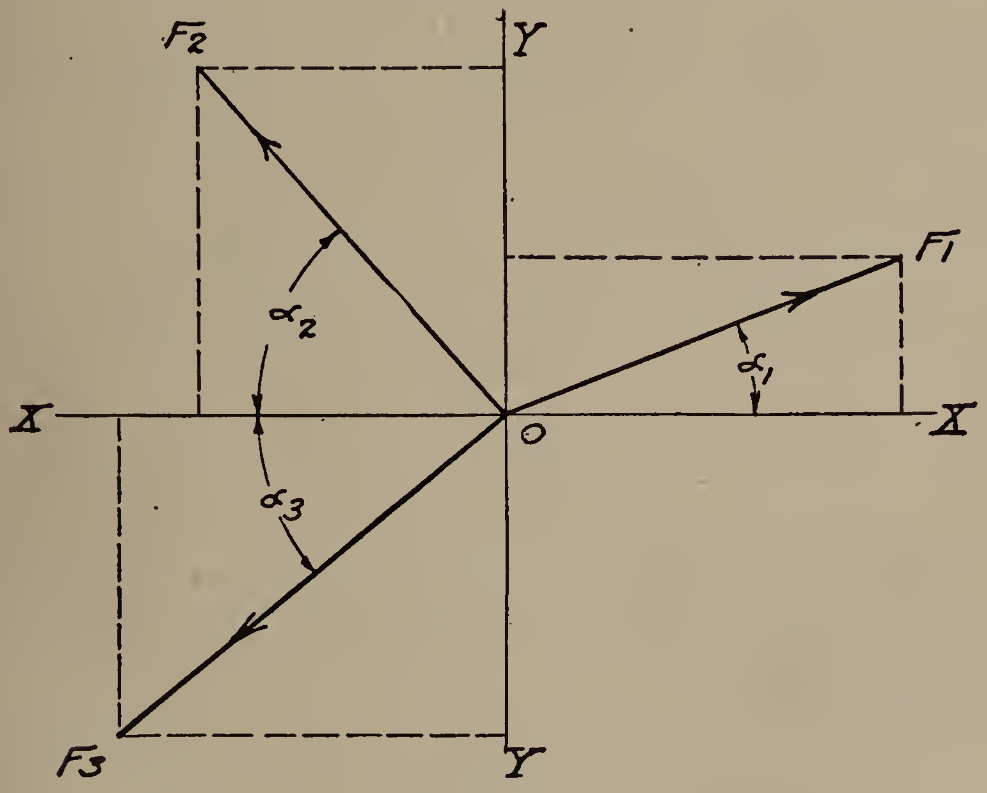


FIG. 7.

Consider the forces acting to the right from the origin O as positive, to the left as negative. Consider forces acting up from the origin as positive and down as negative.

The component of the resultant of the forces along XX will be $+F_1 \cos \alpha_1 - F_2 \cos \alpha_2 - F_3 \cos \alpha_3$ and will act to the right or left as determined by the magnitudes of the components of the forces along this axis.

The component of the resultant of the forces along YY will be $+ F_1 \sin \alpha_1 + F_2 \sin \alpha_2 - F_3 \sin \alpha_3$ and will be up or down as determined by the magnitudes of the components of the forces along this axis.

Assume that the algebraic sum of the components of forces shown in Fig. 7 along XX is positive and represented by Oa (Fig. 8). Assume that the algebraic sum of the components of these same forces along YY is negative and represented by Ob. The resultant R of the forces will be as shown and will have a magnitude $\sqrt{(Oa)^2 + (Ob)^2}$.

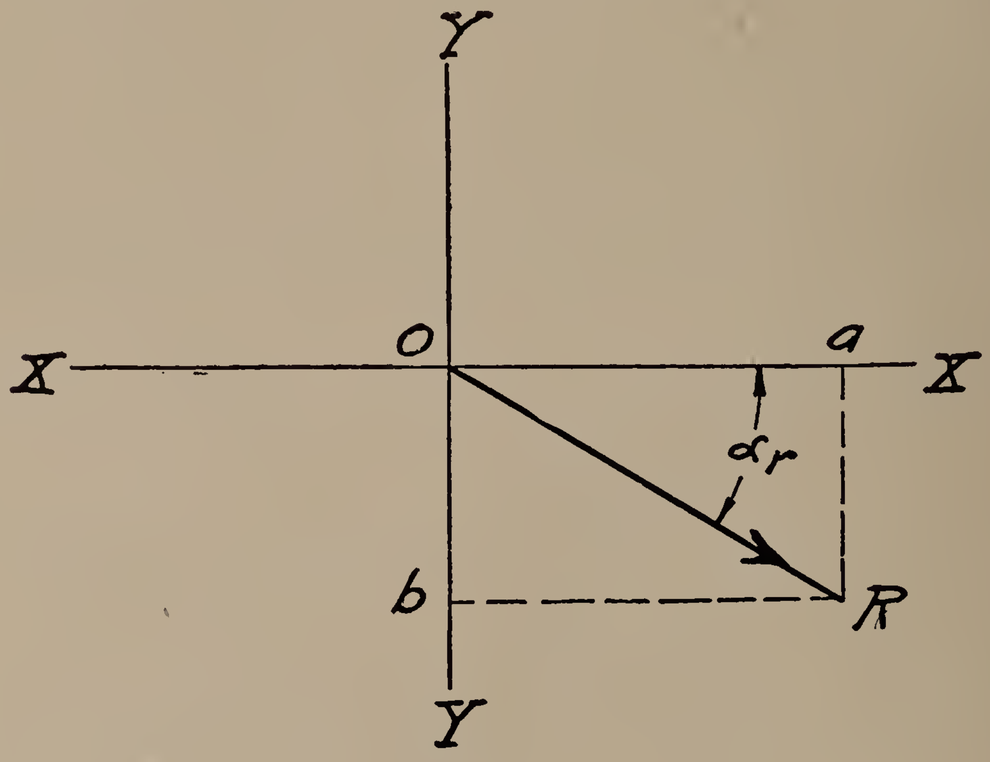


FIG. 8.

The cosine of the angle α_r will be $\dfrac{Oa}{R}$

The sine of the angle α_r will be $\dfrac{Ob}{R}$

The tangent of the angle α_r will be $\dfrac{Ob}{Oa}$

Conditions for Equilibrium. If the forces form a balanced system, the resultant will be zero. In this case, the algebraic sum of the components of the forces along each of the coördinate axes will be zero.

Determination of Unknown Forces. It should be noted that the coördinate axes can be taken in any convenient manner, the origin always being at the point of intersection of the lines of action of the forces.

When the forces form an equilibrium system and some of them are unknown the method of resolution may be used for the determination of the unknown forces.

If nothing is known about the unknown forces, only one can be determined, and this one must be equal and opposite to the resultant of the known forces.

If the lines of action of the unknown forces are known, the method will allow the determination of two unknown forces as follows: (Fig. 9.)

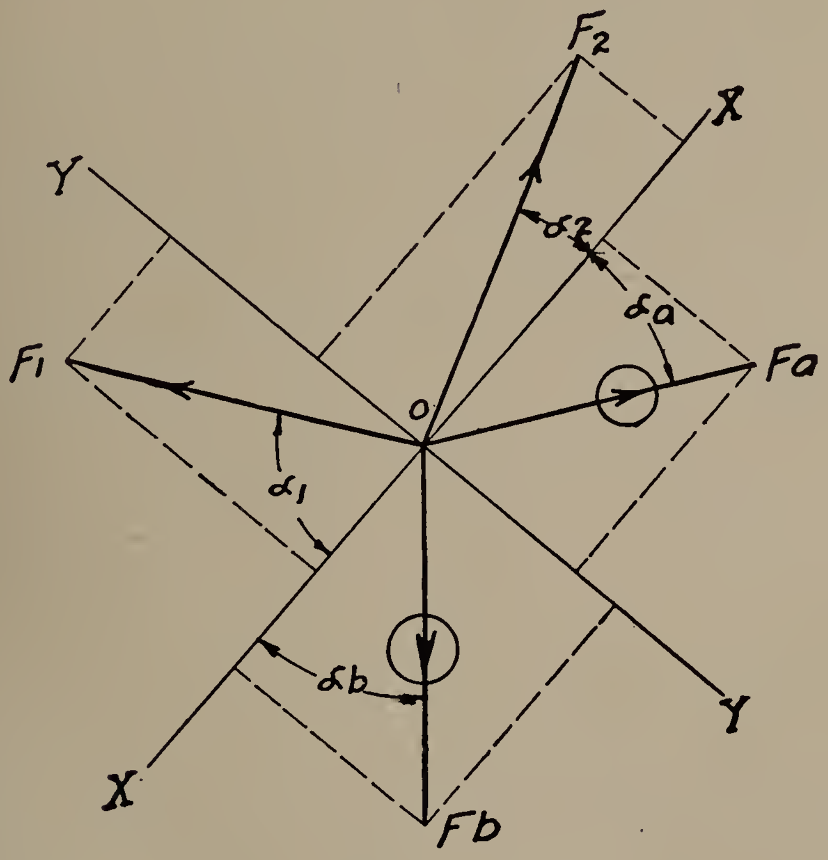


Fig. 9.

F_1 and F_2 are known. F_a and F_b are unknown in magnitude and direction but their lines of action are known and the system is in equilibrium. Select two convenient coördinate axes with an origin at O. The position of these axes will depend upon the manner in which the position of the forces is designated. Let XX and YY be the convenient coördinate axes. First assume directions for F_a and F_b. Let the arrows in the circles be these assumed directions (opposite direction might be assumed just as well). Resolve the forces into components along XX and YY. Add each set algebraically and place equal to zero.

Along XX $F_2 \cos \alpha_2 - F_1 \cos \alpha_1 + F_a \cos \alpha_a - F_b \cos \alpha_b = 0$
Along YY $F_2 \sin \alpha_2 + F_1 \sin \alpha_1 - F_a \sin \alpha_a - F_b \sin \alpha_b = 0$

Solve these two equations for F_a and F_b. If the value of either comes out negative, it simply means that the assumed direction was wrong and the actual direction of the force is opposite from the assumed direction.

If there are more than two unknown forces, the case is statically indeterminate.

Method of Moments. The resultant of a system of forces whose lines of action are in the same plane and pass through the same point, cannot be determined by the method of moments. If the line of action of the resultant is determined by some other method, its magnitude can then be determined by moments by use of the fact that the moment of the resultant of the set of forces with respect to any point in their plane is equal to the algebraic sum of the moments of the forces with respect to the same point.

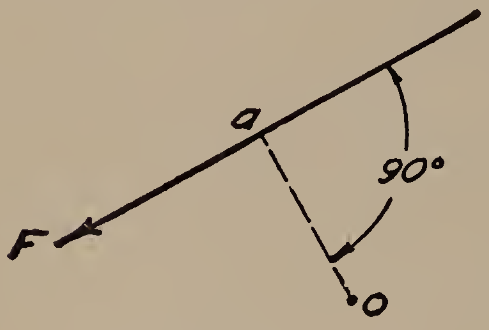


FIG. 10.

The moment of a force with respect to a point is the product of the magnitude of the force and the perpendicular distance from its line of action to the point. (Fig. 10.) The moment of force F with respect to point O is $F \times aO$. The moment in this case is considered to be negative; if the force F acted in the opposite direction its moment about O would be considered to be positive.

Conditions for Equilibrium. When the system of forces is in equilibrium, the algebraic sum of the moments of the forces with respect to any point in their plane must be zero.

Determination of Unknown Forces. If nothing is known about the unknown forces except point of application, no unknown forces can be determined by this method.

If the lines of action of the unknown forces are known, two unknown forces can be determined. (Fig. 11.)

Let F be known and F_a and F_b be unknown in direction and magnitude. Take a moment axis at any convenient point on the line of action of F_b such as x, then $F \times xz - F_a \times xm = 0$.

Therefore $F_a = \dfrac{F \times xz}{xm}$ and must act towards the left as shown by dotted arrow. Take a moment axis at any convenient point on the line of action of F_a, such as y, then

$$-F \times ys + F_b \times yt = 0$$

Therefore
$$F_b = \frac{F \times ys}{yt}$$

and must act to the right as shown by dotted arrow.

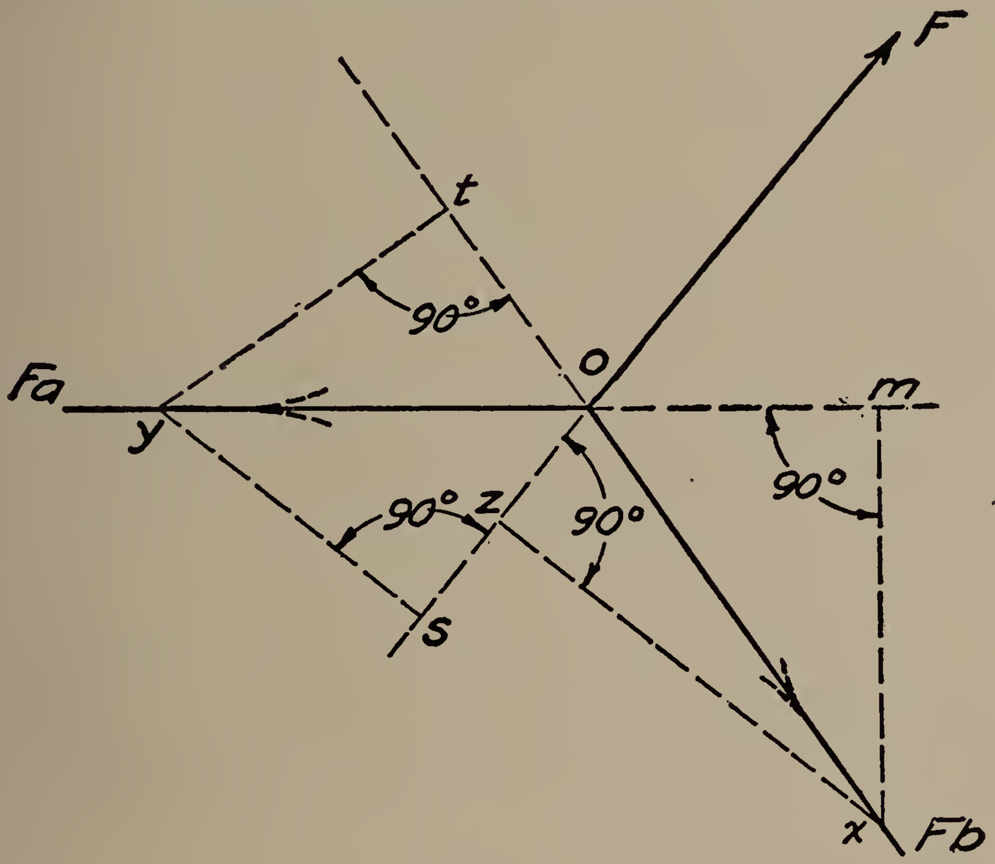


FIG. 11.

If more than two unknown forces are present, the case becomes statically indeterminate.

Care should be exercised in the solution of any problem to use as simple methods as possible. Many problems are very easy of solution by some one of the three methods and at the same time may be rather difficult of solution by one or both of the other methods. Any problem should be carefully analyzed to determine the most convenient method for its solution.

If three forces are in the same plane and are in equilibrium, it follows from the preceding discussion that their lines of action must pass through the same point.

4. Problem. (Fig. 12.) This problem will be solved by all three methods. The frame ABC supports a weight of 1000 pounds as shown; find the stresses in the members AB and AC. Consider the point (in this case the center of the pin) A. The forces acting at this point are the weight W and the forces with which the two members act on the pin. These forces form a system that is in equilibrium. Their lines of action are in the same plane and pass through the same point. Two of these forces are unknown, but their lines of action are known, therefore, the problem is statically determinate.

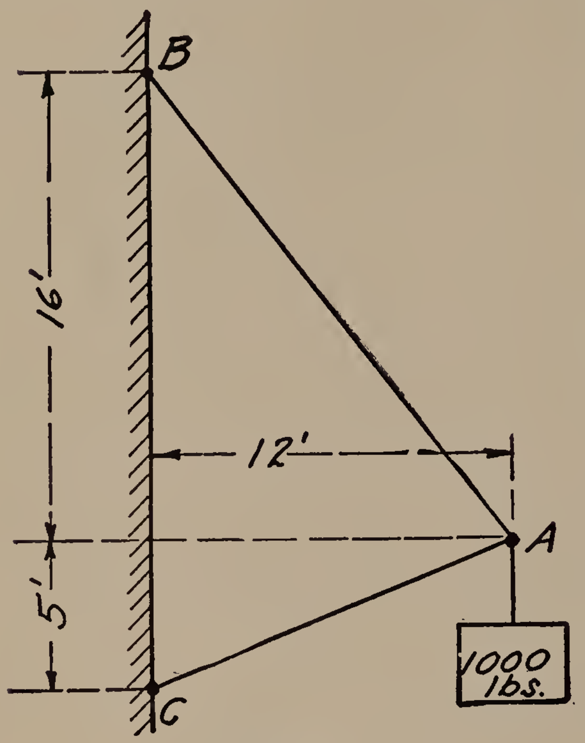


Fig. 12.

The lines of action of the two unknown forces are determined by the fact that the two members AB and AC of the frame are pin connected and are loaded at their ends and their ends only (two force members), in which case the forces exerted by either of them at its ends must be along a line connecting the pins. The structures that will be considered in these notes will be assumed to be pin connected and the weight of the members either neglected or concentrated at the joints.

Stress Polygon Method. (Fig. 13.) Draw bc parallel and proportional to W; the arrow is placed on this line to show how W acts on point A of the frame. From the ends of this line, draw lines parallel to the unknown forces exerted by AB and AC of the frame upon pin A. Two constructions are shown (one is dotted). If this stress diagram had been carefully drawn to scale, the magnitude of F_1 and F_2 could have been taken off directly. The arrows show how these forces act on point A.

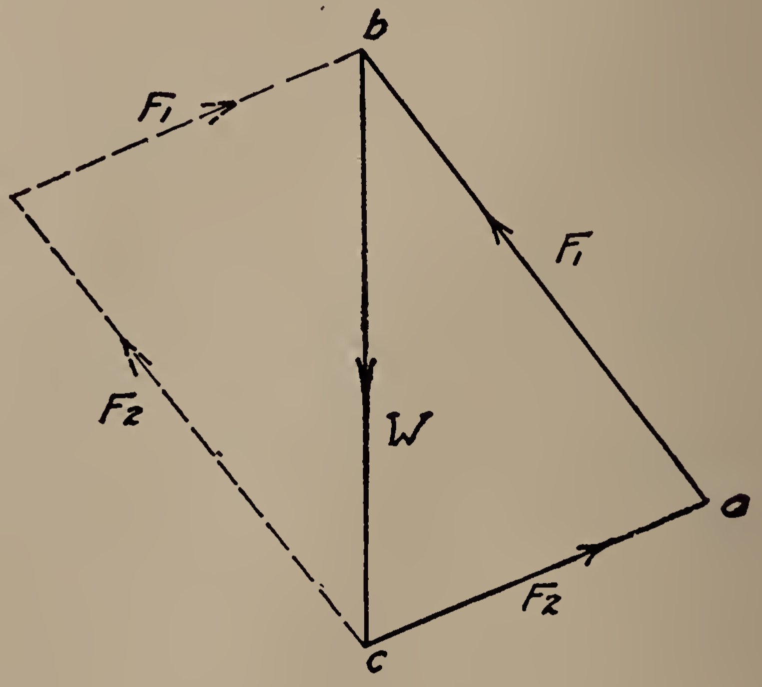


Fig. 13.

The force F_1 acts to the left, showing that the member AB is pulling on the pin and is, therefore, in tension. The force F_2 acts to the right, showing that the member AC is pushing on the pin and is, therefore, in compression.

In order to solve the stress polygon analytically, the relative length of the sides ab, ac and bc must be obtained, then as bc is known to represent 1000 pounds, the forces represented by the other sides can be determined. ab of the stress polygon is parallel to AB of Fig. 12. ca is parallel to CA and bc is parallel to BC. Therefore the triangle abc is similar to the triangle ABC and their homologous sides are proportional.

$$\frac{\sin acb}{\sin bac} = \frac{ab}{bc} = \frac{AB}{BC} = \frac{20}{21} \qquad \text{Therefore } ab = bc \times \frac{20}{21}$$

$$= \frac{1000 \times 20}{21} = 952 \text{ pounds}$$

$$\frac{\sin abc}{\sin bac} = \frac{ac}{bc} = \frac{AC}{BC} = \frac{13}{21} \qquad \text{Therefore } ac = bc \times \frac{13}{21}$$

$$= \frac{1000 \times 13}{21} = 619 \text{ pounds}$$

ab represents the force with which the member AB acts upon the pin A. Therefore stress in AB is 952 pounds tension. The stress in AC is 619 pounds compression.

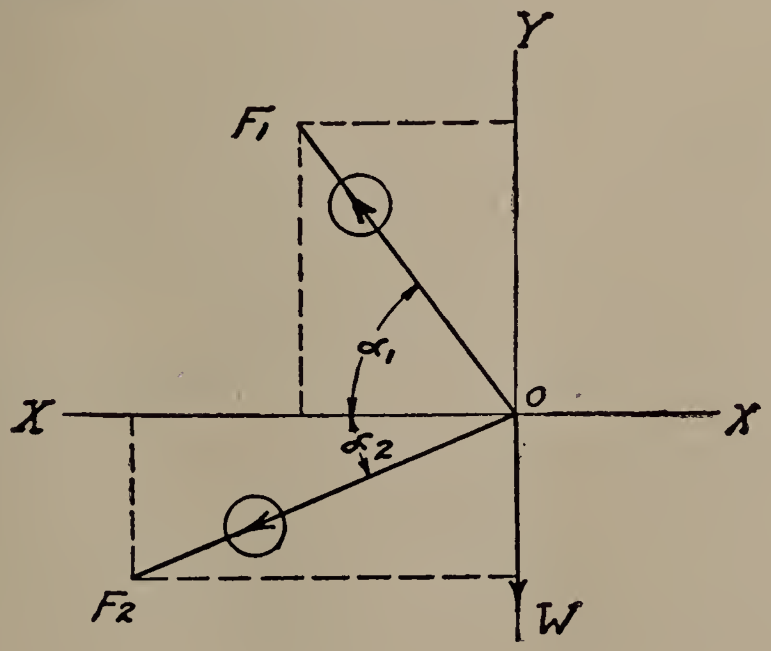


FIG. 14.

Resolution Method. Consider the three forces acting at point A (Fig. 12), and assume the direction of the unknown forces F_1 and F_2 as shown in Fig. 14.

Take as convenient coördinate axes XX and YY. Resolve the forces into components along these axes using the signs as in the previous discussion of the case. Assume directions for F_1 and F_2 as shown. (Arrows in circles.)

$$\sin \alpha_1 = \frac{16}{20} \qquad \sin \alpha_2 = \frac{5}{13}$$

$$\cos \alpha_1 = \frac{12}{20} \qquad \cos \alpha_2 = \frac{12}{13}$$

Force	*Component along XX*	*Component along YY*
1000	0	-1000
F_1	$-\frac{3}{5} F_1$	$\frac{4}{5} F_1$.
F_2	$-\frac{12}{13} F_2$	$-\frac{5}{13} F_2$

Impose the conditions of equilibrium.

$$\text{Components along } XX, \; -\frac{3}{5} F_1 - \frac{12}{13} F_2 = 0$$

$$\text{Components along } YY, \; \frac{4}{5} F_1 - \frac{5}{13} F_2 - 1000 = 0$$

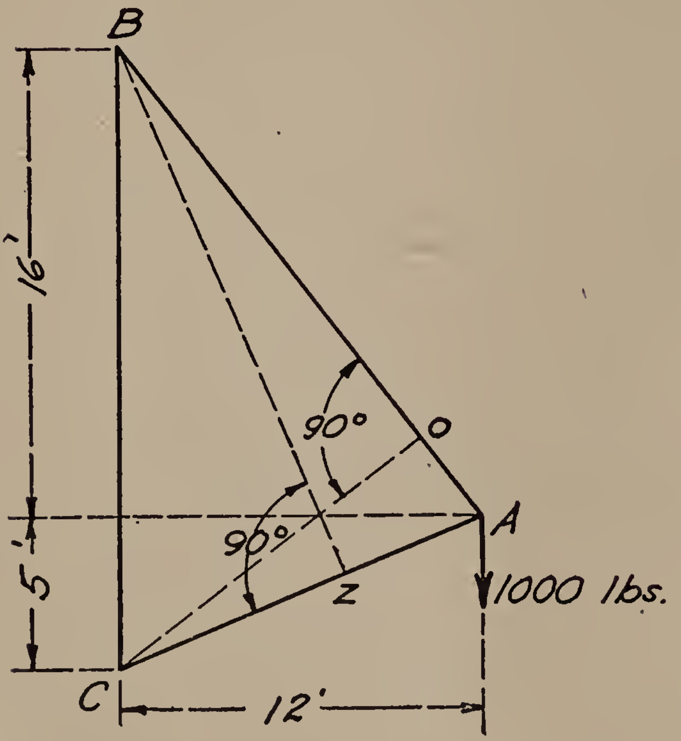


Fig. 15.

Solve for F_1 and F_2, —

$F_1 = 952$ pounds
$F_2 = -619$ pounds

These values show that the direction assumed for F_1 was correct, and that the direction assumed for F_2 was wrong.

Therefore, force exerted by AB on the pin is 952 pounds and member is in tension, and the force exerted by AC is 619 pounds and the member is in compression.

Method of Moments. (Fig. 15.) Take a moment axis at any convenient point on the line of action of F_1, say B. The moment of F_1 about this point is zero. The moment of the 1000 pound force is $1000 \times 12 = 12{,}000$ foot-pounds. This is positive as the moment is right-handed. Impose the condition of equilibrium for this case.

$$\Sigma M = 0$$

$$1000 \times 12 - F_2 \times Bz = 0$$

$$Bz = 19.38 \text{ feet}$$

Therefore $$F_2 = 619 \text{ pounds}$$

and must act to the right upon A as its moment about B is negative.

Take a moment axis on the line of action of F_2 say at C, then

$$1000 \times 12 - F_1 \times oC = 0$$

$$oC = 12.6 \text{ feet}$$

Therefore $$F_1 = 952 \text{ pounds}$$

and must act to the left upon A as its moment about C is negative.

This gives the stress in $AB = 952$ pounds tension and
the stress in $AC = 619$ pounds compression.

5. Parallel Forces in the Same Plane. Formulae for use with parallel forces require for their derivation what may be called, "Couple propositions." Certain of these propositions may be stated as follows:

A couple is two equal and opposite parallel forces. The moment of a couple is the magnitude of either force multiplied by the perpendicular distance between their lines of action. A right-handed couple is designated as positive, a left-handed one as negative.

The moment of the couple (Fig. 16) is right-handed and equal to $F \times ox$. If the forces acted in the opposite direction the moment, while of the same magnitude, would have been negative.

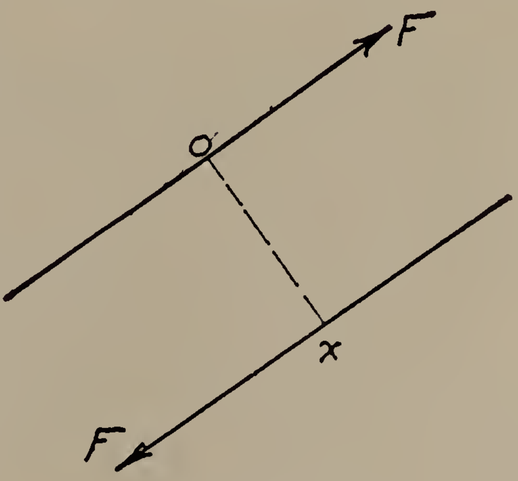


FIG. 16.

The resultant of a number of couples in the same plane is a couple in this plane having a moment equal to the algebraic sum of the moments of the given couples.

A couple may be balanced by any other couple in its plane having a moment equal in magnitude and opposite in sign to that of the given couple.

A couple may be changed in any manner as regards forces, arms, or position in the same plane without altering its effect, provided that the magnitude and sign of its moment remains unchanged.

The resultant of a couple and a single force in the same plane is a single force in the same plane acting parallel and in the same direction as the given force and at a distance from the line of action of the given force equal to the magnitude of the moment of the couple divided by the force. This resultant must be so located that its moment about the line of action of the given force shall have the same sign as the moment of the couple.

Given a couple whose moment is left-handed and equal to 100 foot-pounds and a single force *F* of 25 pounds acting at *A* as shown (Fig. 17). The resultant of this couple and force is a force of 25 pounds parallel to the given force and at a distance from it equal to the moment of the couple divided by the force. This resultant *R* must act as shown by heavy line, because it must have the same direction as *F* and must have a negative moment about the line of action of *F*.

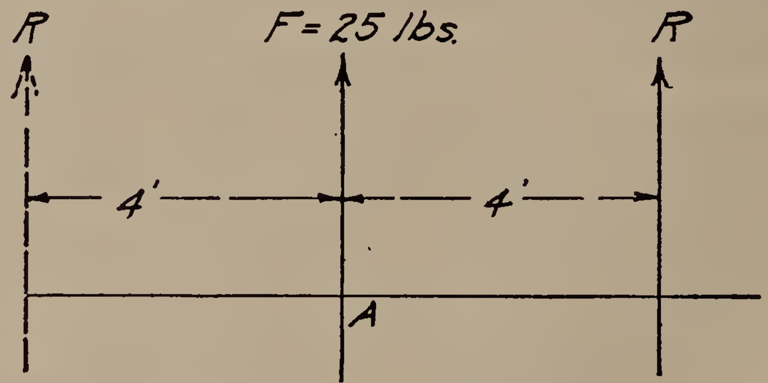


FIG. 17.

If the moment of the couple had been 100 foot-pounds and had been positive, the resultant would have been 25 pounds acting up at a distance of 4 feet to the left of the line of action of *F*, as shown by dotted line.

A single force may be replaced by a couple and a single force as follows:

Let it be required to replace the force *F* (Fig. 18) by a couple and a single force not acting along *F*. $F = 60$ pounds.

Take any couple, say one whose moment is 360 foot-pounds right-handed. The force, that when combined with the couple will have a resultant of *F*, will be a force of 60 pounds, shown full line, acting at a distance of $\frac{360}{60} = 6$ feet to the right of the line of action of F and parallel to it as shown at F_1; therefore, it is possible to replace the force F acting at A by the force F_1 acting at B and a couple in the same plane. F_1 is equal and parallel to F. If it had been desired to replace force $F = 60$ pounds at A by a single force and a left-handed couple whose moment is 360 foot-pounds. The single force would have been $F_2 = 60$ pounds acting as shown by the dotted line at C.

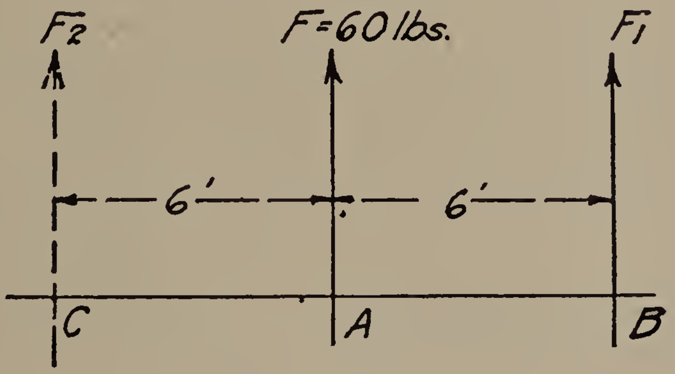


FIG. 18.

The resultant of a system of parallel forces in the same plane is parallel to the forces and equal to their algebraic sum. The point of application of this resultant may be determined by assuming some convenient origin in the plane of the forces and taking moments of the forces about this point. Add the moments algebraically and divide this sum by the algebraic sum of the forces. The quotient will be the distance from the origin to the line of action of the resultant. This distance must be taken in such a direction from the origin that the moment of the resultant will have the same sign with respect to the origin as that of the algebraic sum of the moments of the forces about the same point.

Find the resultant of the three parallel forces shown in Fig. 19. The forces are all in same plane. Choose any point in the plane of the forces as an origin, say O. Let YY be parallel to the forces at O. Replace each of the forces by a single force at O and a couple. The single force at O in each case will be equal and parallel to the force in question and the couple will have a moment whose magnitude and sign is the same as the moment of the force under discussion about the origin.

If each force is treated in the same manner, we shall have at the origin O three forces equal and parallel to the given forces.

The resultant of these three forces will be a single force of $-10 - 40 + 30 = -20$ pounds.

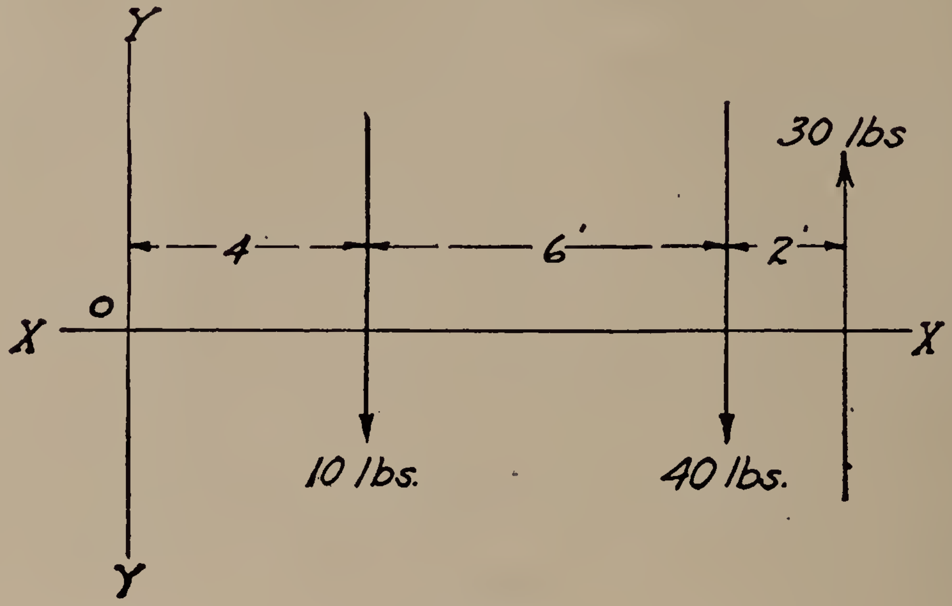


FIG. 19.

The significance of the minus sign is that the force acts down. The forces may also be tabulated as shown below.

Down Forces	*Up Forces*
10	30
40	
50	
30	
20	

The resultant of the three couples will be a single couple whose moment is equal to the algebraic sum of the moments of the three couples.

$$4 \times 10 + 10 \times 40 - 12 \times 30 = +80 \text{ foot-pounds.}$$

The significance of the plus sign is that the moment of the resultant couple is right-handed. The moments of the couples may also be tabulated.

Right-handed Couples	*Left-handed Couples*
4 × 10 = 40	12 × 30 = 360
10 × 40 = 400	
440	
360	
80	

We now have the case of a single force of 20 pounds acting down along YY and a couple in the same plane whose moment is 80 foot-pounds and is right-handed. The resultant of this force and the couple must be a force equal and parallel to the 20 pound force, and acting at a distance from it equal to $80 \div 20 = 4$ feet.

The moment of the final resultant force about O must have the same sign as the moment of the couple. The moment of the couple was right-handed, therefore, the moment of the resultant force about O must be right-handed also. The resultant of the three forces is, therefore, a force of 20 pounds acting down at a distance of four feet to the right of O.

The origin may be taken at any convenient point.

Conditions for Equilibrium. The algebraic sum of the forces must be equal to zero, and the algebraic sum of the moments of the forces about any point in their plane must be equal to zero.

Determination of Unknown Forces. When a system of parallel forces in the same plane is in equilibrium, and nothing is known about the unknown forces, only one unknown force can be determined. This force will be equal and opposite to the resultant of the known forces. When the lines of action of the unknown forces are determined, two unknown forces may be found. (Fig. 20.)

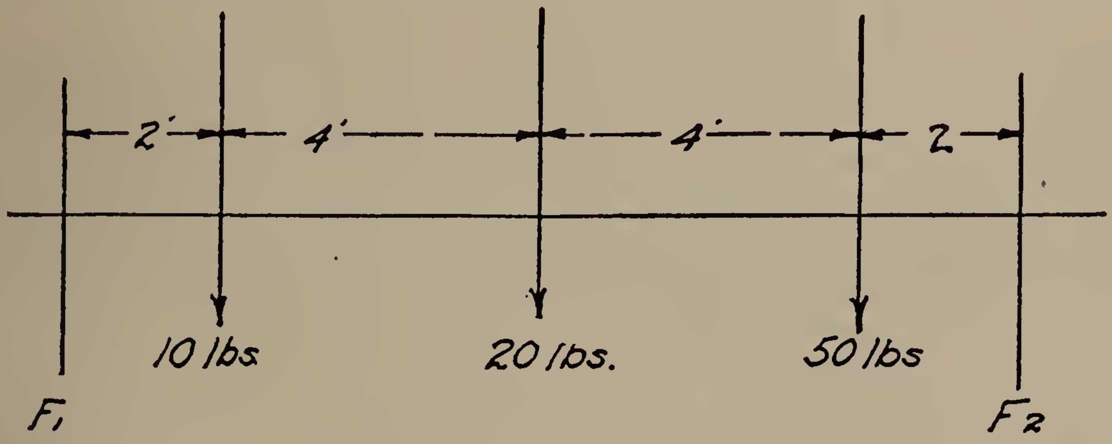


Fig. 20.

This system of forces is in equilibrium, the lines of action of the unknown forces F_1 and F_2 are determined. The conditions of equilibrium are, that the algebraic sum of the forces must be zero and, that the algebraic sum of the moments of the forces about any point in their plane must be zero. Assume a moment axis anywhere on the line of action of one of the unknown forces say F_1. The algebraic sum of the moments of the known forces about this point is $2 \times 10 + 6 \times 20 + 10 \times 50 = +640$ foot-pounds the plus sign indicating a right-handed moment. The algebraic sum of the moments of all the forces acting with respect to the point chosen must be zero, therefore, the moment of F_2 must be 640 foot-pounds and must be left-handed. Therefore F_2 acts up and is equal to $640 \div 12 = 53\frac{1}{3}$ pounds. F_1 may be found by taking an origin anywhere on the line of action of F_2 and making the algebraic sum of the moments zero.

$$-2 \times 50 - 6 \times 20 - 10 \times 10 = -320 \text{ foot-pounds}$$

The minus sign means left-handed. The moment of F_1 must be right-handed, therefore, F_1 must act up and its moment must be 320 foot-pounds. F_1 is equal to $320 \div 12 = 26\frac{2}{3}$ pounds.

These two values may be checked by applying the other test for equilibrium, that the algebraic sum of the forces must be zero.

The Up Forces	*The Down Forces*
$53\frac{1}{3}$	10
$26\frac{2}{3}$	20
80	50
	80

If more than two forces are unknown, even if their lines of action are known, the case becomes statically indeterminate.

6. Uniformly Distributed Forces. A force is uniformly distributed along the line AB (Fig. 21). The intensity of this force

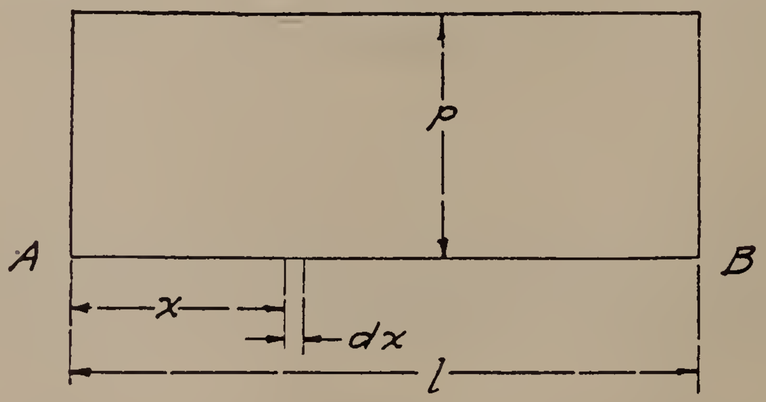


Fig. 21.

is p pounds per foot. Assume an origin at A. Take any elementary length of the line AB such as dx at distance x from A. The force acting on this length is pdx, and the whole force R is $\int_0^l pdx =$ pl pounds. To find the point of application of this resultant, take moments of all the elementary forces about A and divide by R. Let x_0 equal distance from A to the point of application of R.

$$x_0 = \frac{\int_0^l px\,dx}{\int_0^l pdx} = \frac{l}{2} \text{ (to the right of } A\text{)}$$

When a force is uniformly distributed, the resultant is equal to the intensity of the force multiplied by the length of the line over which it is distributed and the point of application of this resultant is at the center of this line.

7. Uniformly Varying Forces. A force varies uniformly in intensity from A to B. (Fig. 22.) The intensity at B is n pounds per foot. a is the rate of change in intensity per foot of length and is also the intensity at distance unity from A. a in this case $= \frac{n}{l}$. The intensity at distance x from A is $ax = \frac{n}{l}x$. The elementary force acting upon length dx is $pdx = axdx$.

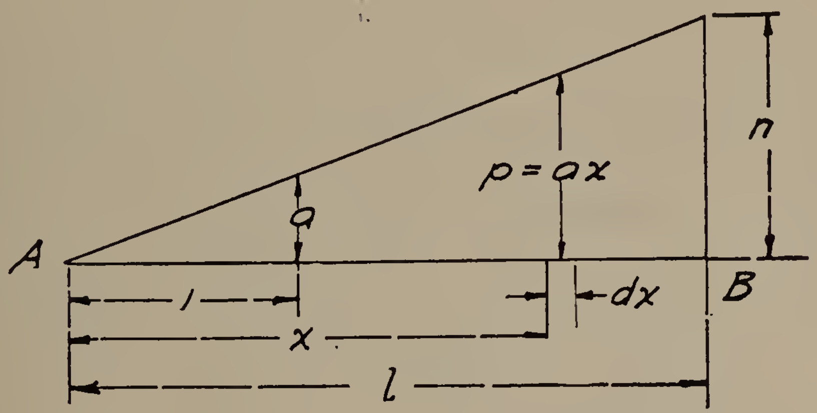


FIG. 22.

Resultant R is equal $\int_0^l ax\,dx = \frac{al^2}{2}$ but $a = \frac{n}{l}$

Therefore $$R = \frac{nl^2}{2l} = \frac{nl}{2}$$

The distance from A to the point of application of R is x_0.

$$x_0 = \frac{\int_0^l ax^2dx}{\int_0^l ax\,dx} = \frac{2\,l}{3} \text{ (to the right of A)}$$

When a force is uniformly varying from zero along a line, the resultant is equal to one-half of the maximum intensity multiplied by the length of the line over which the force is distributed and the point of application of this resultant is at a distance from the point of zero intensity equal to two-thirds of the length of the line.

Find the resultant of the force distributed along line *AB* as shown in Fig. 23 being 100 pounds per foot at *A* and 500 pounds per foot at *B* and uniformly varying.

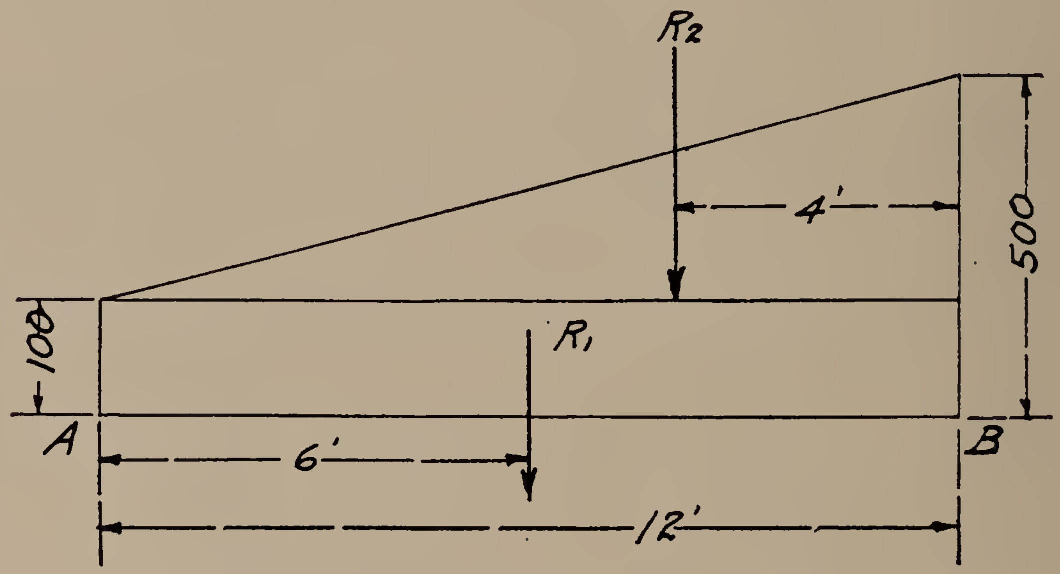


FIG. 23.

Consider the force to be divided into two parts, a uniformly distributed force of an intensity of 100 pounds per foot, and a force of uniformly varying intensity from zero at *A* to 400 pounds per foot at *B*.

The resultant of the uniformly distributed force is 1200 pounds and acts at R_1. The resultant of the uniformly varying force is 2400 pounds and acts at R_2. The resultant R of the two forces is 3600 pounds and its point of application can be found by taking moments: —

Take moment axis at *A*.

$$1200 \times 6 + 2400 \times 8 = 3600 \times x_0$$

Therefore x_0 is $7\frac{1}{3}$ feet measured to the right from *A*.

The beam shown (Fig. 24) is loaded with a uniformly distributed load of 100 pounds per foot of length. A uniformly distributed load of 200 pounds per foot from *B* to *D*, and a uniformly varying force from zero at *C* to 200 pounds per foot at *A*. The beam is supported at the ends; find the supporting forces R_A and R_B.

Take moments about the point of application of the left-hand support, and use the resultants of the different distributed forces.

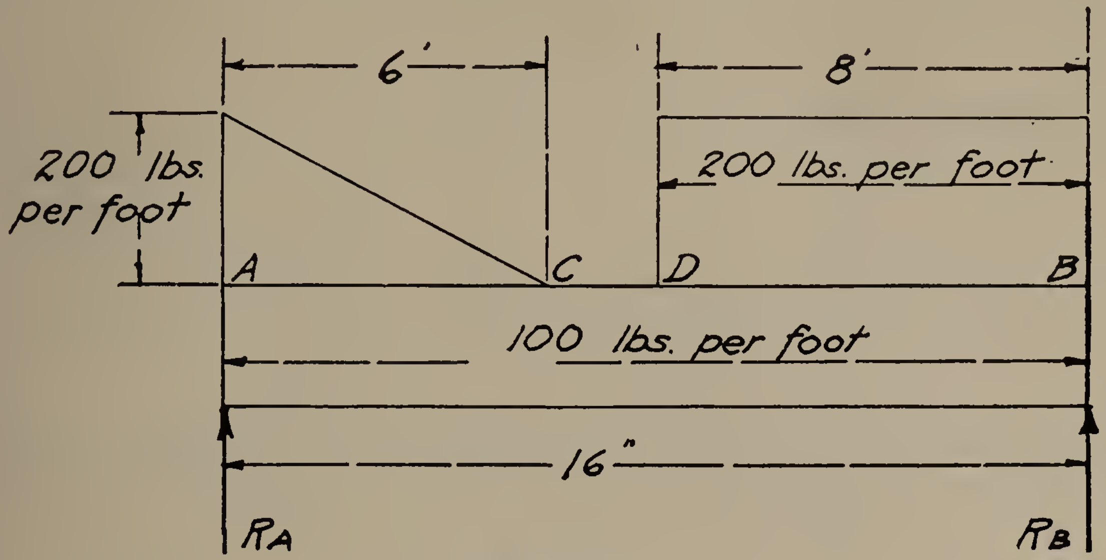


FIG. 24.

Apply the condition that the algebraic sum of the moments of all the forces about A must be zero.

$$\begin{array}{rr}
\frac{200}{2} \times 6 \times 2 = & 1200 \\
100 \times 16 \times 8 = & 12{,}800 \\
200 \times 8 \times 12 = & \underline{19{,}200} \\
 & +\,33{,}200
\end{array}$$

Therefore, $33{,}200 - 16\,R_B = 0$

Therefore, $R_B = 2075$ pounds (up)

Take moments about point of application of right-hand support.

$$\begin{array}{rr}
200 \times 8 \times 4 = & 6400 \\
100 \times 16 \times 8 = & 12{,}800 \\
\frac{200}{2} \times 6 \times 14 = & \underline{8400} \\
 & -\,27{,}600
\end{array}$$

$-27{,}600 + 16\,R_A = 0$

Therefore, $R_A = 1725$ pounds (up)

Check by using the condition for equilibrium that the algebraic sum of all the forces must be zero.

$$\frac{200}{2} \times 6 + 100 \times 16 + 200 \times 8 = 3800 \text{ down.}$$

$$2075 + 1725 = 3800 \text{ up}$$

Therefore, $\Sigma\,V = 0$

8. Varying Forces in General. A vertical load is distributed along AB (Fig. 25) in such a manner that the intensity at any point x distant from A is always x^2 pounds per foot.

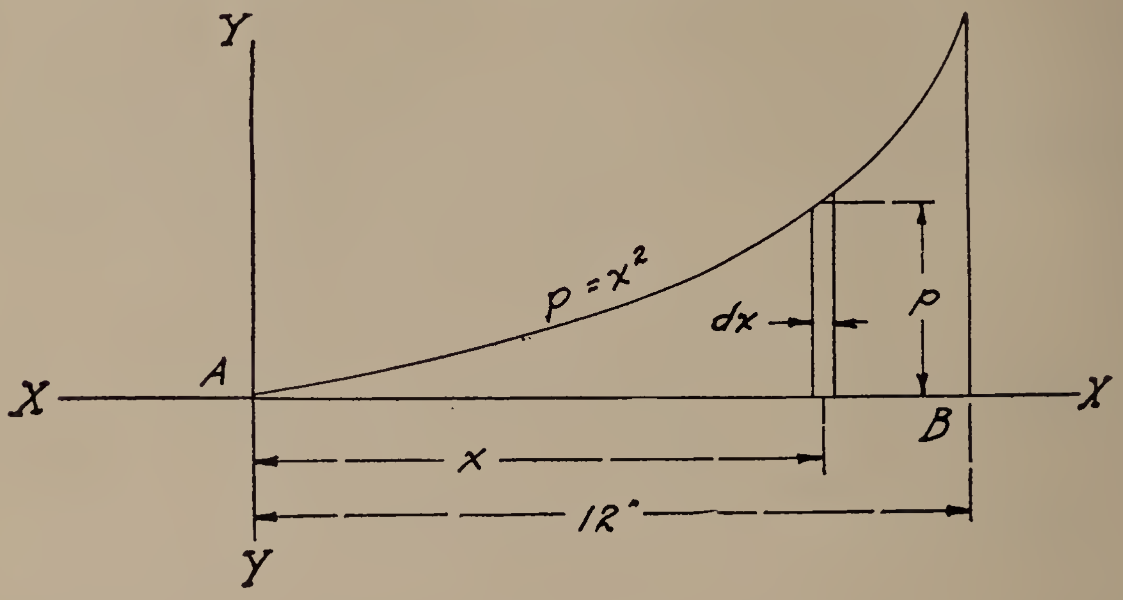


FIG. 25.

The elementary force is $pdx = x^2\,dx$.

$$\text{Therefore} \qquad \text{resultant} \int_0^{12} x^2\,dx = 576 \text{ pounds.}$$

$$\text{Take moments about } A \text{ then } x_0 = \int_0^{12} x^3\,dx \div 576 = 9 \text{ feet.}$$

Resultant is 576 pounds acting down at a distance of 9 feet to right of A.

Assume this load to be balanced by two vertical forces at A and B.

$$\frac{576 \times 9}{12} = R_B \qquad \text{Therefore } R_B = 432 \text{ pounds up}$$

$$576 - 432 = R_A \qquad \text{Therefore } R_A = 144 \text{ pounds up}$$

9. General Case of Forces Acting Anywhere in the Same Plane. The general case of forces acting anywhere in the same plane covers the preceding, where the lines of action of the forces are either parallel or pass through the same point. Therefore, in the following discussion, systems of forces that do not come under these two cases, will be considered.

The method of handling such cases can be outlined as follows: Assume a convenient origin and two convenient coördinate axes in the plane of the forces. Resolve the forces into components parallel to these axes. Replace each of these components by a force equal and parallel to it at the origin and a couple, the moment of which will be the moment of the component about the origin. Combine the forces at the origin into a single resultant force. Combine the couples into a single resulting couple; then determine the final resultant by combining the single force at the origin with the resultant couple.

Care should be taken to select an origin and coördinate axes that will make the solution of the particular problem as simple as possible.

Find the resultant of the three forces shown in Fig. 26. Choose the origin, in this case *O*. Choose coördinate axes, in this case *XX* and *YY*. It should be clearly understood that the axes need not be horizontal and vertical. The conditions of each problem will determine the best origin and axes.

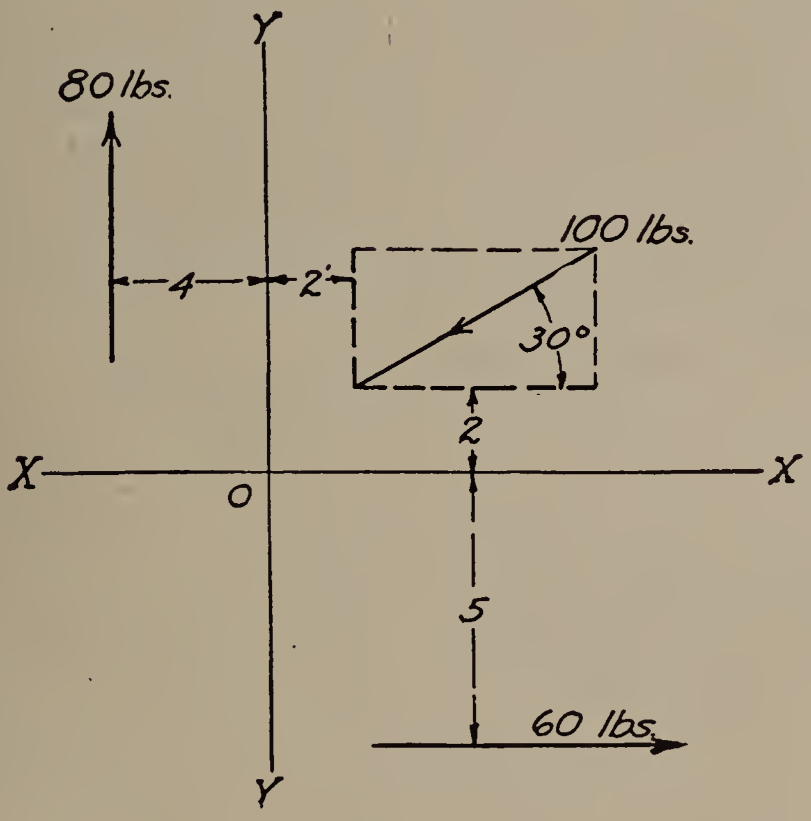


FIG. 26.

Resolve the forces into components along *XX* and *YY*. It will be noted that the 80 pound force is parallel to *YY* and that the 60 pound force is parallel to *XX*.

The *XX* component of the 100 pounds force is 86.6 pounds.

The *YY* component of the 100 pounds force is 50 pounds.

Replace each of the components or forces that does not act along one of the coördinate axes by a single force equal and parallel to it at the origin and a couple.

First consider these forces at the origin. (Fig. 27.)

These forces are all in the same plane and act through the same point. Their resultant may be determined by any one of the three methods available, stress polygon, resolution, or moments. Use the resolution method in this case. It will be noted that the forces are already resolved along two axes at right angles to each other.

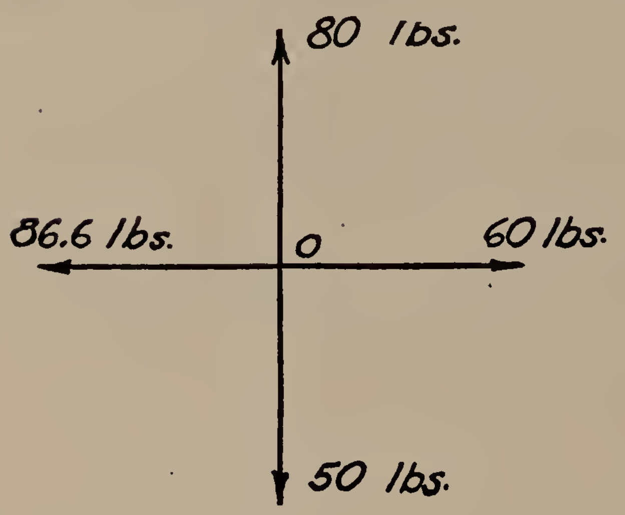


Fig. 27.

The algebraic sum of the forces acting along YY is $-50 + 80 = +30$. 30 pounds acting up. The algebraic sum of the forces acting along XX is $60 - 86.6 = -26.6$. 26.6 pounds acting to the left. The resultant of the forces must be

$$\sqrt{(30)^2 + (26.6)^2} = 40 \text{ pounds.}$$

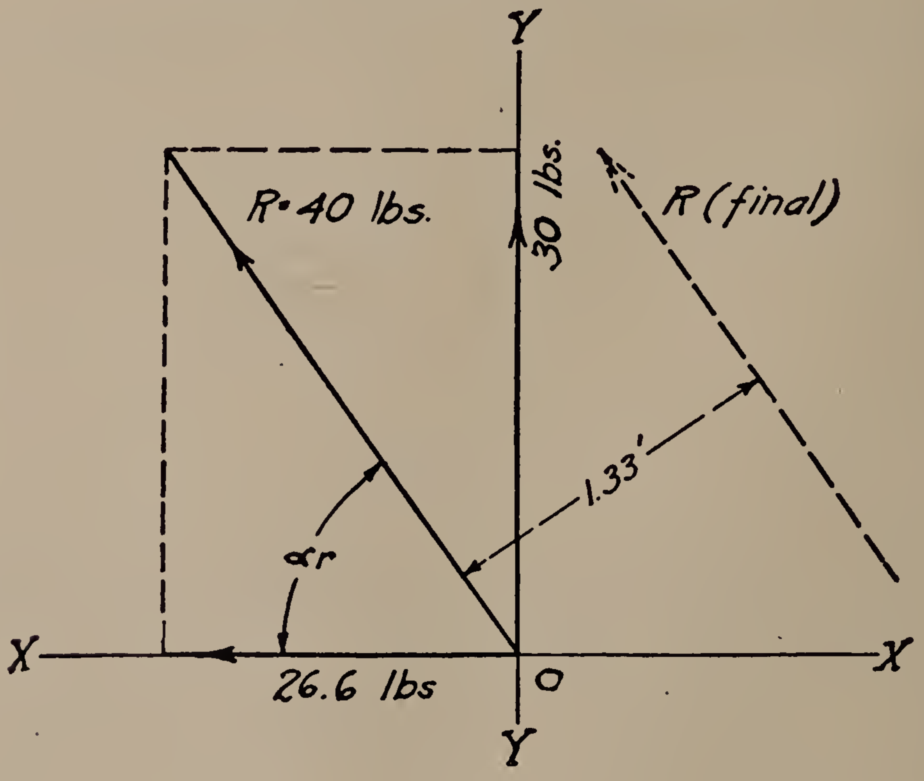


Fig. 28.

The line of action of the resultant can best be determined by the method indicated in Fig. 28.

Lay off from O, 30 pounds up and 26.6 pounds to the left. R

will show line of action of the resultant. The angle α_r can be found by observing that its sine is 30 ÷ 40 = .750. This angle is, therefore, about $48\frac{2}{3}$ degrees.

Now combine the couples. (Fig. 26.)

Right-handed Couples	*Left-handed Couples*
4 × 80 = 320	5 × 60 = 300
2 × 50 = 100	2 × 86.6 = 173.2
420	473.2
	420
	53.2

The resultant of the couples is a couple whose moment is 53.2 foot-pounds and left-handed. The final resultant of the system is the resultant of the force of 40 pounds acting as shown (Fig. 28) and a left-handed couple whose moment is 53.2 foot pounds. This final resultant is then a force of 40 pounds acting parallel to R at the origin, full line (Fig. 28) and at a distance from it equal to 53.2 ÷ 40 = 1.33 feet. The moment of this force about O must be left-handed. The final resultant of the system is shown dotted (Fig. 28) and marked R (final).

Conditions for Equilibrium. The conditions for equilibrium for the general case of forces in the same plane are as follows: The algebraic sum of the moments of the forces about any point in their plane must be zero, and the algebraic sum of the components of the forces along any two coördinate axes in the plane of the forces must be zero.

The conditions are usually expressed

$$\Sigma M = 0$$
$$\Sigma F_x = 0 \qquad \Sigma \text{ (sigma) meaning algebraic sum.}$$
$$\Sigma F_y = 0$$

These conditions for equilibrium may also be stated: — a stress polygon representing the forces must close and ΣM must equal zero.

Determination of Unknown Forces. When the system of forces is in equilibrium and nothing is known about the unknown forces, only one unknown force can be determined. This force must be equal and opposite to the resultant of the known forces.

When the lines of action of the unknown forces are determined, three unknown forces can be found if the lines of action of the forces are not parallel or do not pass through the same point.

If there are more than three unknown forces, the case becomes statically indeterminate.

The majority of the problems met with in structural design will come under the general case of forces acting in the same plane.

10. Frames. The determination of the supporting forces for a frame or of the stresses in any of its members is made possible by the condition that the frame itself or any part of it must be held in equilibrium by the forces acting externally to the frame as a whole or to the part under consideration. The following problems will illustrate their determination. When the supporting forces for a structure are under discussion, the loads may be considered to be either distributed or concentrated, but when the stresses in the members are being determined the loads must be distributed at the joints of the frame.

In all the problems taken up here, it will be assumed that the frames do not deflect appreciably under the loads, that the members are pinned together, and that the loads are all applied at the joints.

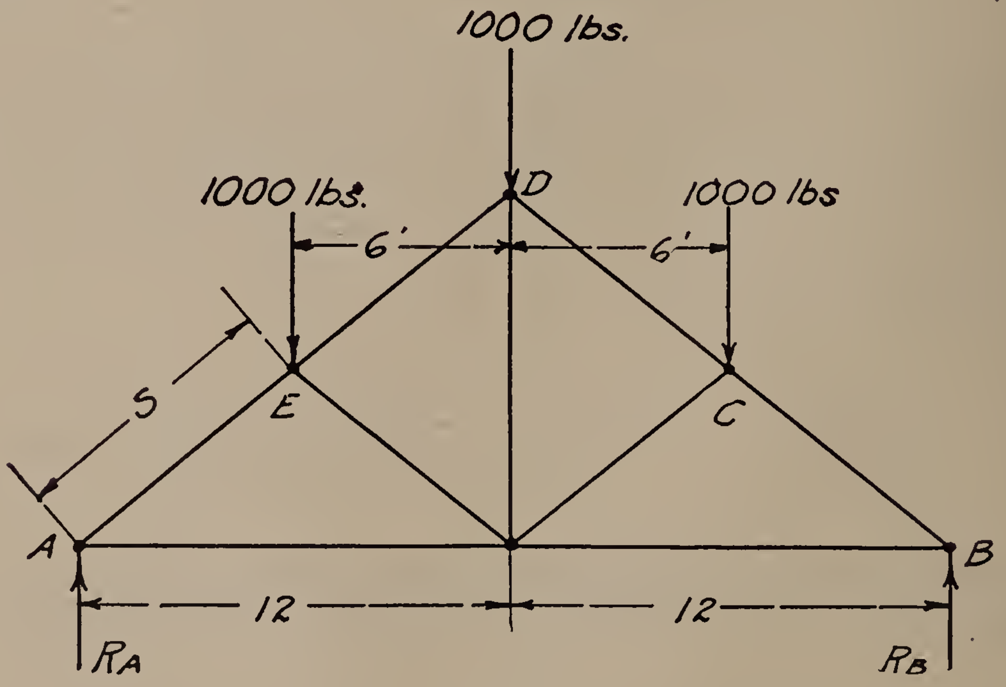


Fig. 29.

The frame shown (Fig. 29) is loaded at the joints E, D and C with 1000 pounds. Find the supporting forces at A and B.

As the frame is practically rigid as a whole and the loads are vertical, the supporting forces at A and B will also be vertical.

The forces acting will all lie in the same plane. Make use of the conditions of equilibrium for this case,

$$\Sigma M = 0 \qquad \Sigma F_x = 0 \qquad \Sigma F_y = 0$$

Take moments about A.

$$\begin{aligned} 1000 \times 6 &= 6000 \\ 1000 \times 12 &= 12000 \\ 1000 \times 18 &= \underline{18000} \\ & 36000 \text{ (right-handed).} \end{aligned}$$

$$36000 - 24\,R_b = 0$$

Therefore $R_b = 1500$ pounds (up)

$$1000 + 1000 + 1000 = 1500 + R_a$$

Therefore $R_a = 1500$ pounds (up)

The loads could have been considered to be concentrated at D in which case: —

$$3000 \times 12 = 36000$$

$$36000 - 24\,R_b = 0$$

Therefore $R_b = 1500$ pounds (up)

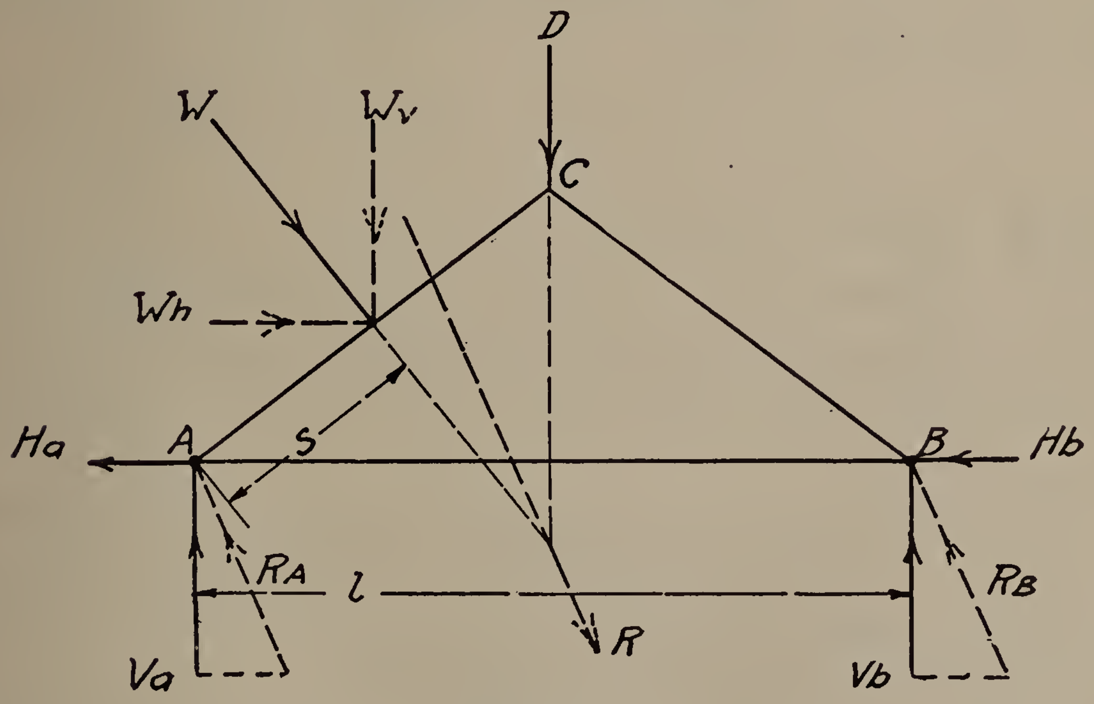


Fig. 30.

It should be noted carefully that the method using the resultant of the loads can only be used when supporting forces only are to be determined.

The symmetrical frame (Fig. 30) is loaded with a vertical dead load and a wind load on left. The supporting forces are applied at A and B. The frame is designed in such a manner as to be practically rigid. D is taken as the resultant of the dead load and W as the resultant of the wind load acting at right angles to AC at its center.

(1) Consider the supports at A and B to be pin connected. This case will be statically indeterminate as nothing is known about the supporting forces except their points of application. The more convenient way of looking at this case is to consider all of the forces applied to the frame to be resolved into H and V components. If this is done, there will be four unknown forces of which the lines of action are known. This is also a statically indeterminate condition. V_a and V_b can be determined but not H_a and H_b.

(2) Consider the case where the line of action of one of the supports is known. Assume that the left end of the frame is pinned and that the right end rests on rollers. H_b will be zero and there will be three unknown forces, whose lines of action are neither parallel nor pass through the same point.

Take moments about A.

$Ws + (Dl \div 2) - V_b l = 0$. Therefore V_B acts up and is equal to $[Ws + (Dl \div 2)] \div l$.

$$V_a + V_b - D - W_v = 0$$

Therefore V_a acts up and is equal to $D + W_v - V_b$

$H_b = 0$. Therefore $H_a = W_h$ and acts towards the left. If the roller is assumed to be at A and B is pinned, V_a and V_b will be the same as before but H_a will be zero and H_b will be equal and opposite to W_h.

It will be noted that as long as the frame and the loads remain the same, the vertical components of the supporting forces will be the same, while the horizontal components of the supports will depend upon the method of supporting.

(3) Assume the frame to be pinned at A and B and that the horizontal components of the supporting forces are equal. V_a and V_b are the same as above. $H_a = H_b = W_h \div 2$.

(4) Another assumption that is sometimes made is that the supports are parallel to the resultant load in this case V_a and V_b remain unchanged. H_a and H_b may be determined from the

triangles shown in Fig. 30, where R_a and R_b are parallel to the resultant load R.

$$V_a \div H_a = V_b \div H_b \text{ and } H_a + H_b = W_h.$$

H_a and H_b can be obtained by solving these equations. The make up of the frame will have no effect upon the supporting forces unless considerable variations in weight occur.

It will be noted that the frame shown in Fig. 30 will keep its shape and requires no supports other than those necessary to balance the loads.

Consider a frame such as that shown in Fig. 31. This frame must have the supporting forces at A and B in order that it shall be at all stable and both of the ends must be fastened.

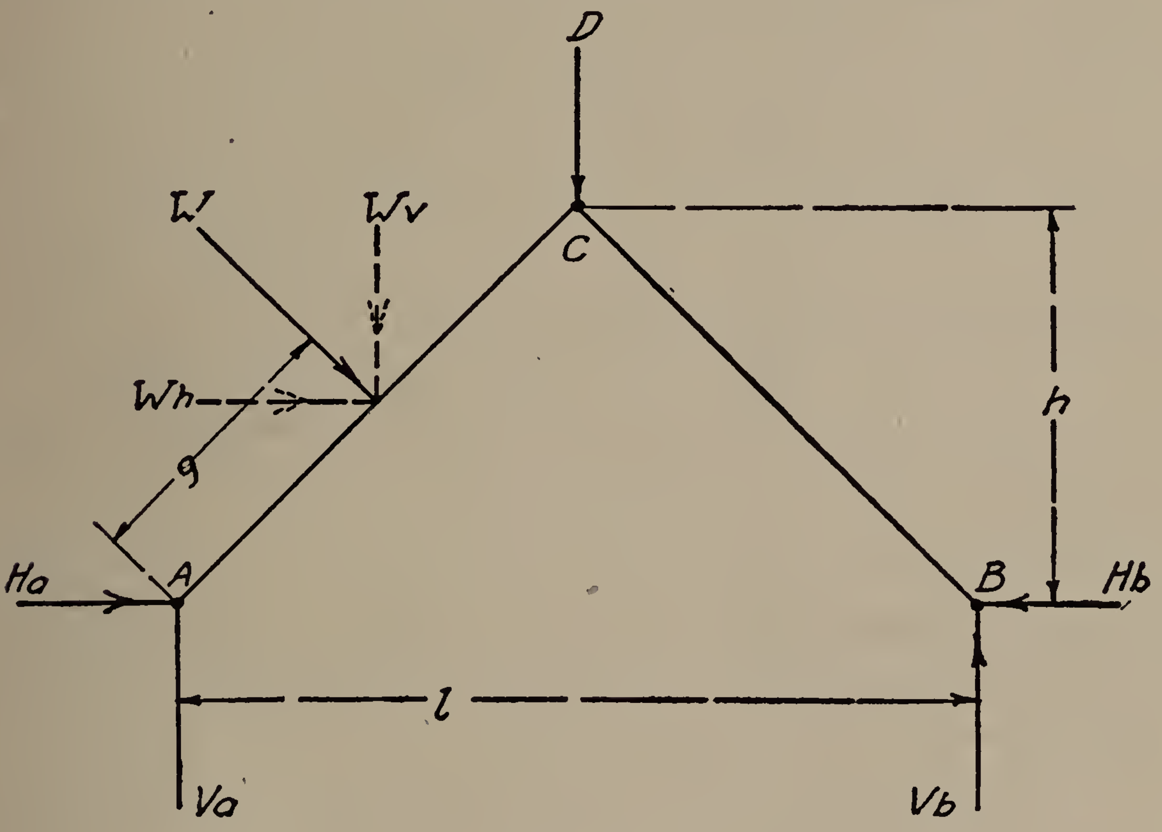


Fig. 31.

V_a and V_b can be determined as in the preceding problem by taking a moment axis at A and using the conditions of equilibrium for the forces acting upon the whole frame $\Sigma M = 0$ and $\Sigma V = 0$.

In order to determine the H components, the frame must be taken in parts. Take the part of the frame CB alone. This part of the frame is acted upon by whatever forces are applied by the pin at C and the supporting force at B. Impose the conditions of equilibrium for these forces $\Sigma M =$ zero taking moments about C, then $- V_b \times \frac{l}{2} + H_b \times h = 0$

Therefore H_b must act to the left and be equal to $V_b \times \frac{l}{2} \div h$.

H_a can be found in the same way by taking the part of the frame AC and imposing the condition ΣM = zero for the forces acting on this part. A check can be made by imposing the condition ΣH = zero for the whole frame.

If the stresses in the members of a frame are to be determined, the forces must be distributed where they belong and the design of the frame must be known completely.

Consider the truss (Fig. 32). First determine the supports R_a and R_e. As the truss is practically rigid as a whole there are no horizontal components required at the supports as the loads are all vertical. Take moments about A.

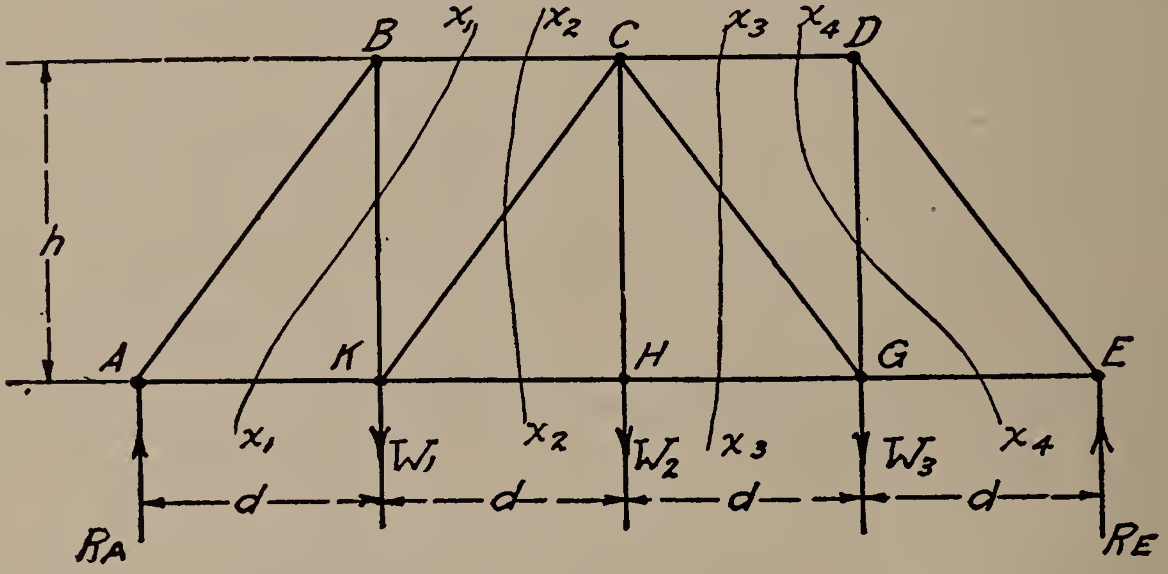


FIG. 32.

$$W_1 d + W_2 \times 2d + W_3 \times 3d - R_e \times 4d = 0$$

Therefore,

R_e acts up and is equal to $W_1 \times d + W_2 \times 2d + W_3 \times 3d \div 4d$.
R_a can be determined by taking a moment axis at E or by $\Sigma V = 0$

Down	*Up*
$W_1 + W_2 + W_3$	$R_A + R_e$.

Therefore R_A acts up and is equal to $W_1 + W_2 + W_3 - R_e$.

The stresses in the members can be determined by what is known as the method of joints. In this method, any joint of the frame is chosen where there are only two unknown forces. The lines of action of all the forces are known because the members are loaded only at the ends and they are assumed to be pinned together.

Start at either point A or E. Impose the conditions of equilibrium for forces at a point in the same plane.

$$\Sigma H = 0 \qquad \Sigma V = 0$$

The vertical component of the force exerted by AB must be equal and opposite to R_a as the force exerted by the horizontal member AK can have no vertical component. Therefore stress in AB must be equal to $R_a \times \frac{AB}{h}$ and must act down on A. This member is, therefore, in compression.

The horizontal component of the force exerted by AB must be balanced by the force exerted by AK. Therefore stress in AK must be the stress in AB multiplied by $AK \div AB$ and must act on point A to the right. AK is, therefore, in tension.

The point E may be treated in the same manner and the stresses in DE and GE determined.

Pass to point B or D. Only two unknown forces now act on either of these points. Then take points G or K and then C and H. It is possible to start at one end of the truss and work from point to point across to the other end. This method will fail if the truss is so constructed as to show more than two unknown forces at every available point at any stage of the solution. Note that the stress in CH can be determined by inspection using point H.

The truss can also be solved by what is known as the method of sections as follows:

Determine the supporting forces, then determine the stresses in AB, AK, DE and GE in the same manner as before.

Now make use of the very important fact, that when a truss is in equilibrium as a whole, any part of it must also be in equilibrium under the forces acting externally to that part.

Assume a plane x_1x_1 cutting the truss into two parts and consider the part on the left of the section. The forces acting externally to that part are R_a and the forces exerted by the members BC, BK and AK. These forces are all in the same plane and their lines of action are known. Impose the conditions of equilibrium $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$. From $\Sigma V = 0$, the stress in BK can be obtained. This stress must be equal and opposite to R_a because BC and AK have no vertical components. Therefore, $BK = R_a$ and is tension because BK must pull down on the part of the truss under consideration to prevent a motion up due to R_a.

From $\Sigma H = 0$, BC must be equal and opposite to AK. AK has been found and is tension, therefore, BC must be of equal magnitude and compression.

Pass plane x_2x_2 through the truss and consider the part to the left of this section. The external forces acting on this part are R_a, W_1, and the stresses in BC, KC and KH.

From $\Sigma V = 0$, the stress in KC can be determined because its vertical component must be equal and opposite to the algebraic sum of R and W_1. BC has been already determined, therefore, the stress in KH can be obtained by imposing the condition $\Sigma H = 0$. KH can also be determined by $\Sigma M = 0$. Take point C as a moment axis, then the moment of stress KH about this point must be equal and opposite to the algebraic sum of the moments of R_a and W_1 about the same point. The stresses in the other members of the truss can be determined by making use of the planes x_3x_3 and x_4x_4. It should be noted that the part of the truss on either side of the plane may be used and that it makes no difference where the planes are taken or what their shape.

A general rule for taking planes may be stated as follows: The plane should not cut more than three members. Not more than two of these members may be parallel or pass through the same point and none of the members can be loaded at more than two points.

Many truss problems can be most easily solved by graphical methods.

11. Graphical Statics. Simple graphical methods for the solution of trusses make use of the stress polygon for forces in a plane and upon the fact that when three forces in the same plane are in equilibrium, their lines of action must pass through the same point.

Supporting Forces (Fig. 33). The frame shown (Fig. 33) is supported at A and B and is loaded at C and E as shown. The line of action of the resultant of W and D must pass through O which is the intersection of the lines of action of W and D. The magnitude and direction of this resultant can be determined graphically by the triangle of forces (Fig. 34). W and D are drawn parallel to the loads on the frame and proportional to them to some convenient scale. R is the magnitude of the resultant, the arrow showing its direction through O (Fig. 33). Draw a line through O parallel to R (Fig. 34). This line will be the line of action of the resultant load. If there were more than two loads, the resultant of any two of them could be combined with a third, and this resultant with a fourth force and so on until the final single resultant of all the loads is found.

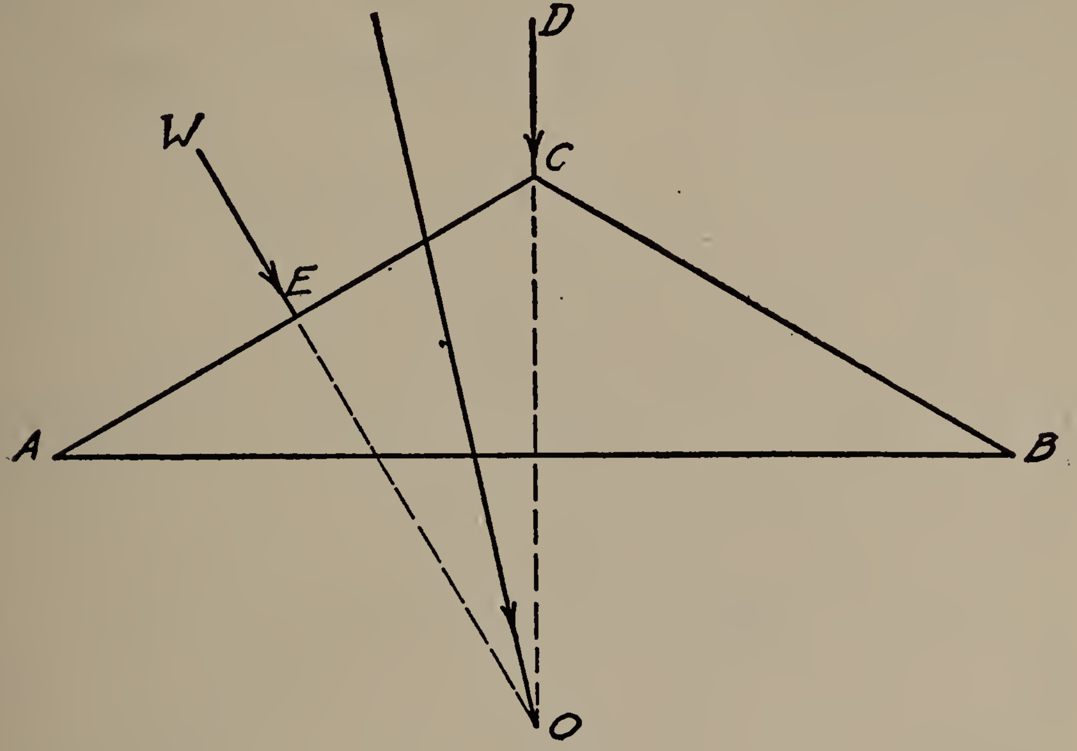


FIG. 33.

Let R (Fig. 35) be the resultant of all the loads on a frame which is supported at A and B. Before the supporting forces can be determined, some assumption must be made concerning them as the general case is statically indeterminate.

(1) Assume a roller at A no friction.

(2) Assume a roller at B no friction.

(3) Assume H components of supports to be equal.

(4) Assume supports parallel to resultant load.

(5) Assume roller at one end with friction.

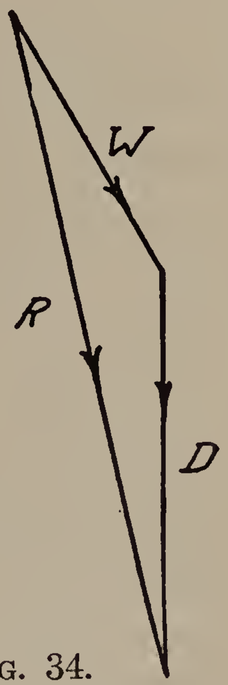


FIG. 34.

Case (1). The support at A will be vertical. Find the point O where its line of action intersects the line of action of R. The line of action of the support at B must pass through O as the three forces are in the same plane and are in equilibrium. Draw a stress polygon for the three forces (Fig. 36).

xy is magnitude of vertical support at A. yp is magnitude and direction of support at B. Both to the same scale as R.

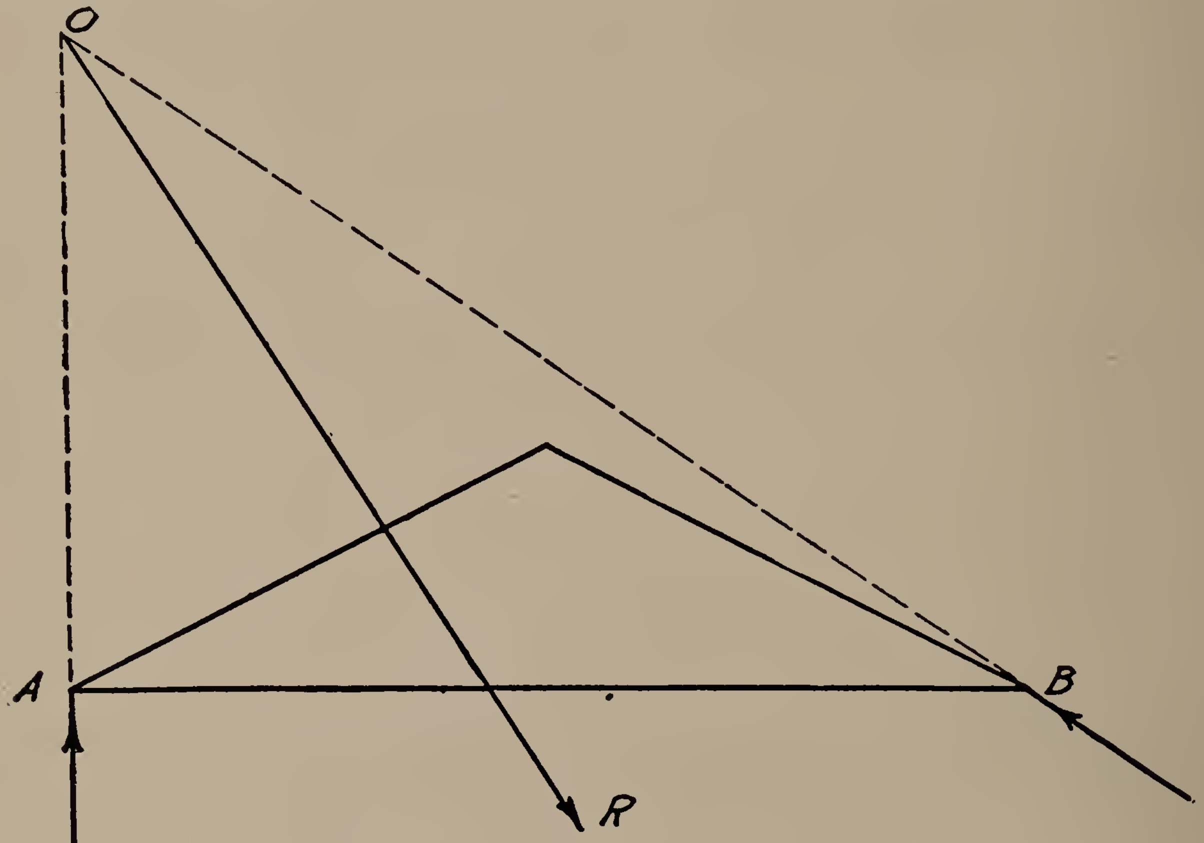


FIG. 35.

Conditions 2–3–4–5 can be applied to the same diagram by making use of the fact that the vertical components of the supporting forces do not depend upon the method of supporting. xy will always be the vertical component of the supporting force at A. yz will always be the vertical component of the supporting force at B.

Case (2). The supporting force at B must be vertical. Resolve R into horizontal and vertical components. pz is the horizontal component. Draw the line ym parallel to pz. ym is equal to the horizontal component of the loads. mp must be the support at B and xm is the support at A.

Case (3). Bisect the line my at s. my is the whole horizontal component of the loading, therefore, ys which equals sm is one half of this component, sp is support at B, sx is the support at A.

Case (4). The supports must be parallel to R and the sum of these horizontal components must be my. wp is the support at B. wx is the support at A.

Case (5). Assume the roller at *B*. Multiply *mp* by the coefficient of friction and lay this value off as *mk*. *kp* is support at *B*. *kx* is support at *A*. If the roller is at *A*, *xt* is one support; *tp* is the other. The supporting forces can be determined without regard to the exact make up of the frame, provided the loads are located. When the stresses in the members are to be found, the truss must be drawn completely.

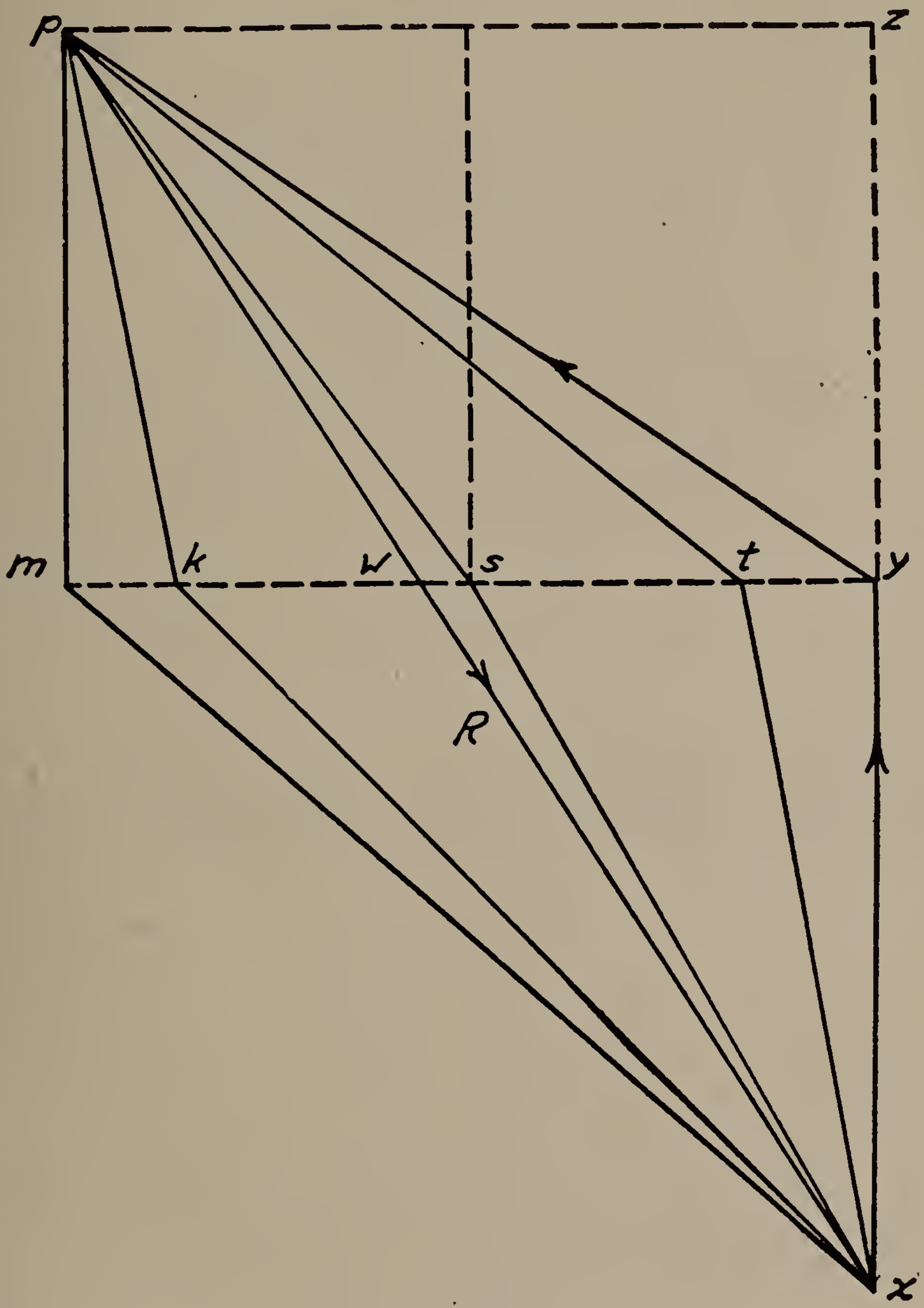


FIG. 36.

The truss (Fig. 37) is loaded with vertical loads as shown and supported at the ends. The supports can be found and separate stress polygons drawn for the joints progressively starting wherever there are only two unknown forces. This method will require a number of stress polygons. In order to simplify the diagram, Bow's Notation is used.

Bow's Notation (Fig. 38). The forces F_1, F_2, F_3 and F_4 all act

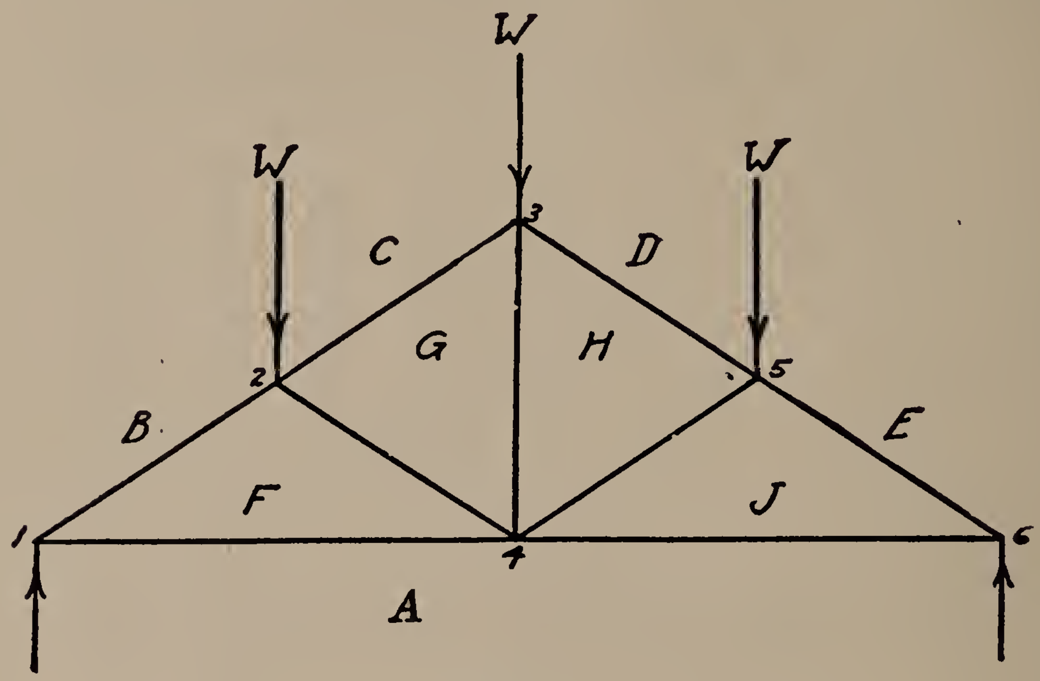


Fig. 37.

in the same plane and at the same point, and form a system of forces that are in equilibrium. Place the letters A, B, C and D

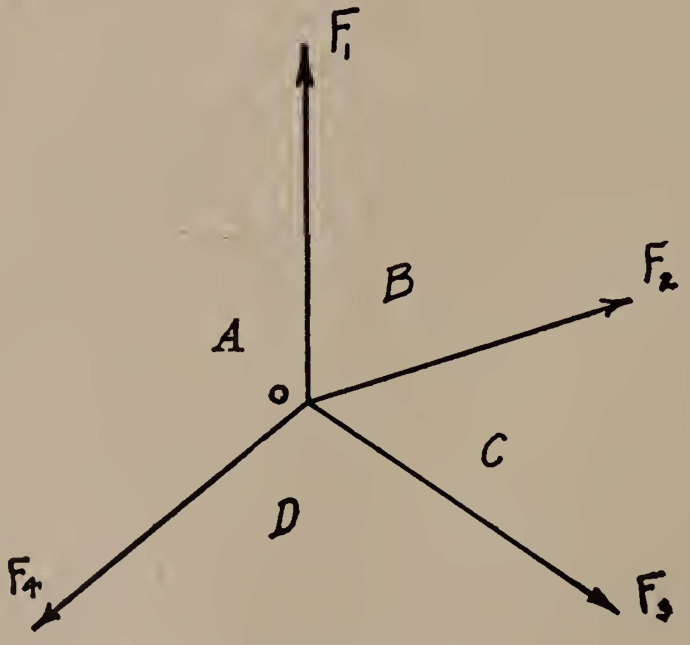


Fig. 38.

in such a manner that each force comes between two letters. Start with any force and construct a stress polygon taking the forces in order, right-handed. (Fig. 39.) In this case start with F_1. Letter the line drawn parallel and proportional to this force in such a manner that reading the letters that appear on each side of the force (Fig. 38) right-handed with respect to the point O will indicate the direction in which the force acts upon the point of intersection of the forces.

Start with a and move towards b (Fig. 39). This shows the way in which the force F_1 acts upon the point O. From b draw bc parallel and proportional to F_2. Read the letters that appear on each side of F_2 (Fig. 38) right-handed, and follow the same order in Fig. 39 and obtain the direction of F_2 with respect to the point.

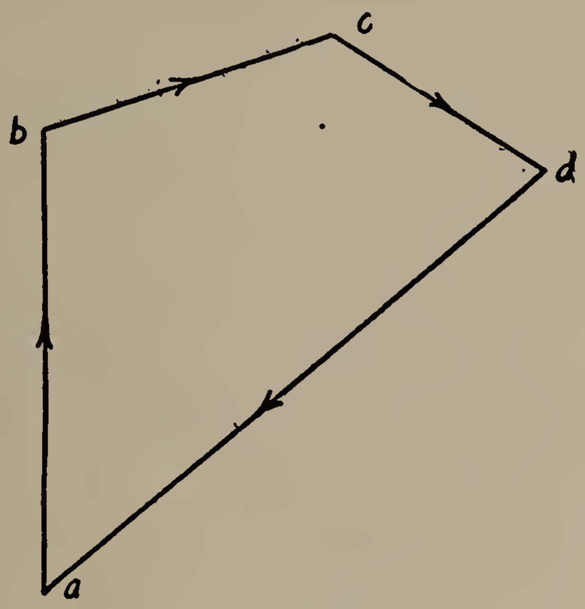


Fig. 39.

Carry out the same operation with F_3 and F_4. The arrows in the stress polygon must follow each other in the same general direction.

Note that if the letters representing any force (Fig. 39) are read right-handed on Fig. 38, the direction of the force with respect to the point is determined from Fig. 39.

Place letters on the truss drawing (Fig. 37) in such a manner that all loads, supports and members will come between two letters. The forces will be designated by the letters between which they are located, reading the letters right-handed with respect to the joint upon which the forces are considered to act, i.e., take the joint at left end of the truss; there are three forces acting at this point. The support will be AB as it comes between A and B. AB is the order of the letters reading right-handed about the joint. BF will be the force in the slanting member of the truss and FA will be the force in the horizontal member.

When a graphical solution of a truss is to be made, the polygon of external forces acting on the truss as a whole is usually drawn first. This polygon for the truss (Fig. 37) is shown by Fig. 40 and is *bcdeab*.

Begin with the left-hand load and proceed right-handed around the truss. *ea* and *ab* are the equal supporting forces. Small letters are generally used in the stress diagram. Start at left end of the truss at joint 1 (Fig. 37). The force *ab* appears on the stress diagram as drawn for the external forces. From *b* draw a line parallel to *BF* and from *a* draw a line parallel to *FA*. Mark the point where they meet *f*. The length of these lines will represent the magnitude of the two forces to the same scale that the external force polygon does the loads and supports. To determine whether the stresses are tension or compression, note that reading the letters, right-handed around the joint *AB* designates the supporting force. Starting with *a* on the stress polygon, and moving towards *b* indicates the direction of the force with respect to the joint. Start with *b* and move towards *f*, this will indicate the direction of the stress in *BF*, with respect to the point. It will be seen that this force is acting towards the joint and must, therefore, be compression. *fa* on the stress polygon shows that the stress *FA* is away from the joint and, therefore, is tension. Consider joint 2. Now that *FB* is known, there are only two

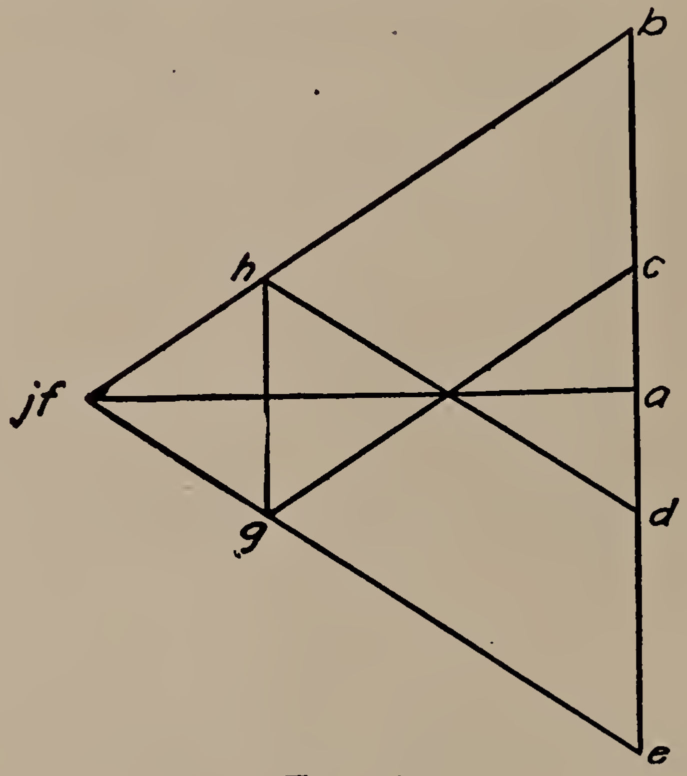


Fig. 40.

unknown forces acting on the joint. Read the letters right-handed around this joint. Stress *FB* and load *W* are already in the diagram (Fig. 40). *fb* taken in order indicates that the stress in *FB* acts upward and is pushing on the joint. Therefore, *FB* is compression as before. *bc* shows that *W* acts down. *cg* indicates that *CG* acts down and is in compression. *gf* acts to the left, showing *GF* to be in compression. The stress diagram may be completed by passing from joint to joint.

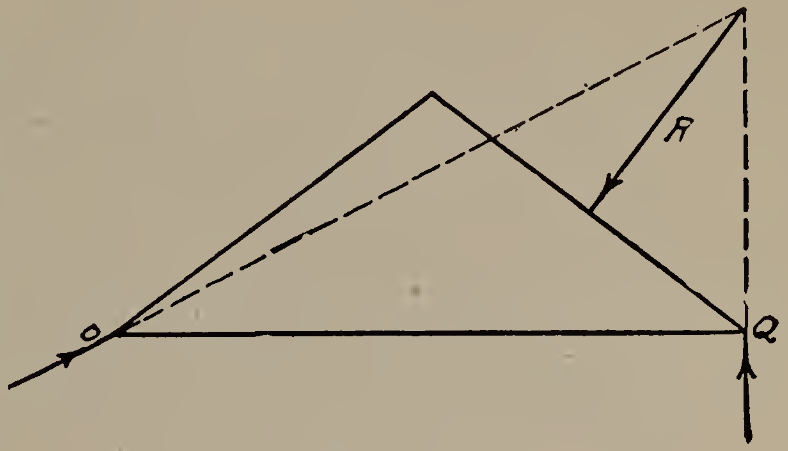


Fig. 41*a*

The stresses in the members of a truss can always be determined graphically by means of the stress polygon and Bow's Notation if no joint is met with where there are more than two unknown forces, or no member that is loaded at any other point than at the two pins.

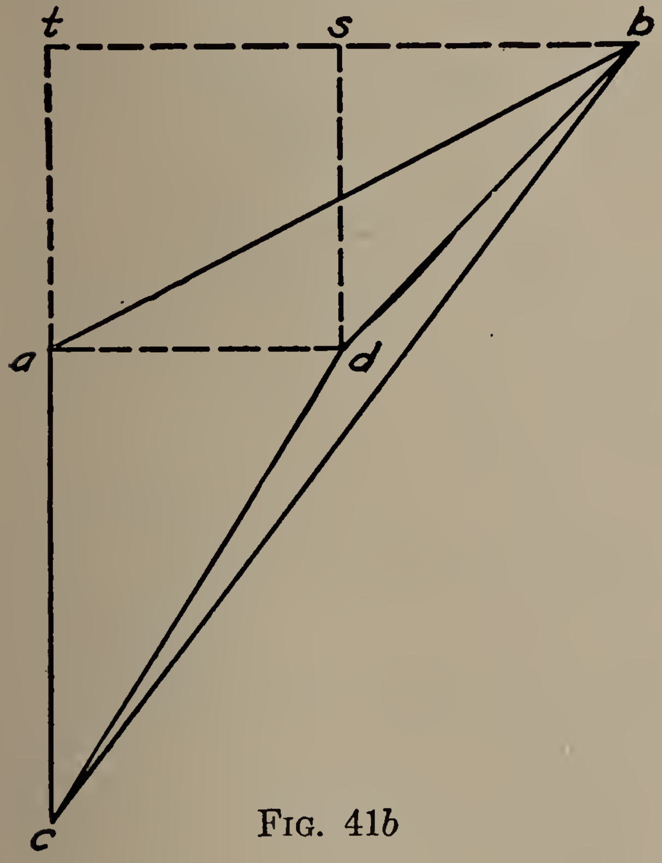


Fig. 41*b*

Wind Loads. The wind is always considered to act at right angles to the surface. A stress diagram for wind on the right is shown in Figs. 41*a* and 41*b*. The horizontal components of the supports are assumed to be equal. Fig. 41*a* is the frame and *R* is the resultant of the wind load. Assume a roller at *Q*, then the line of action of the supports will be as shown. Construct a stress diagram for the forces.

bc is the resultant wind load.

ca is support at *Q*.

ab is support at *O*.

Make use of the fact that the vertical components of the supporting forces are independent of the method of supporting, and make *ad* = *st* = *sb*. Then the supporting forces for the condition of equal horizontal components are *cd* at *Q* and *db* at *O*.

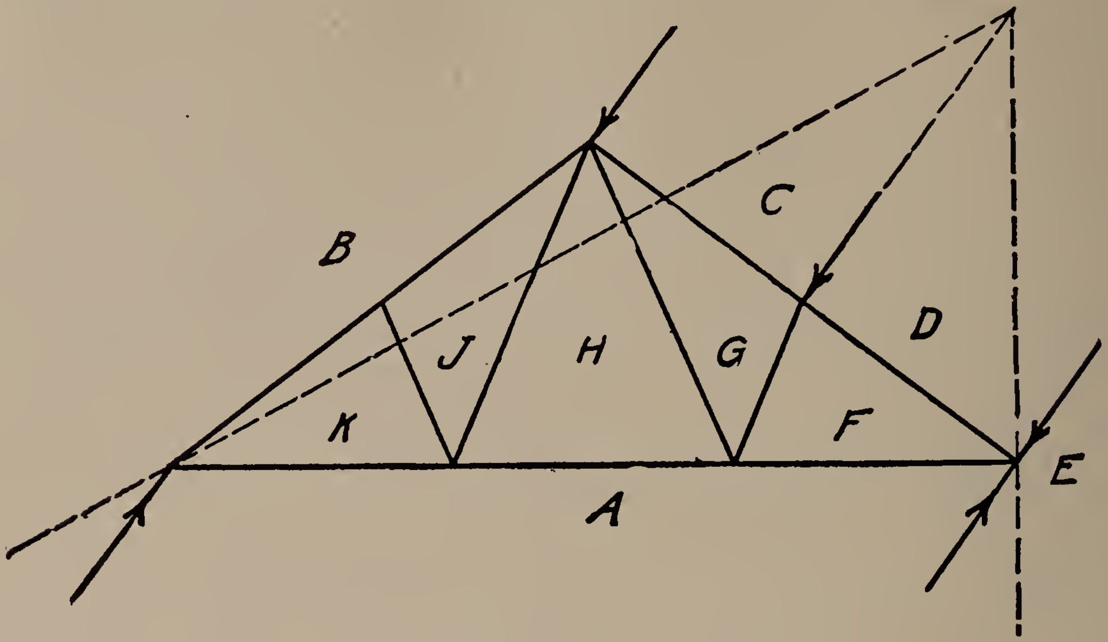


Fig. 42*a*

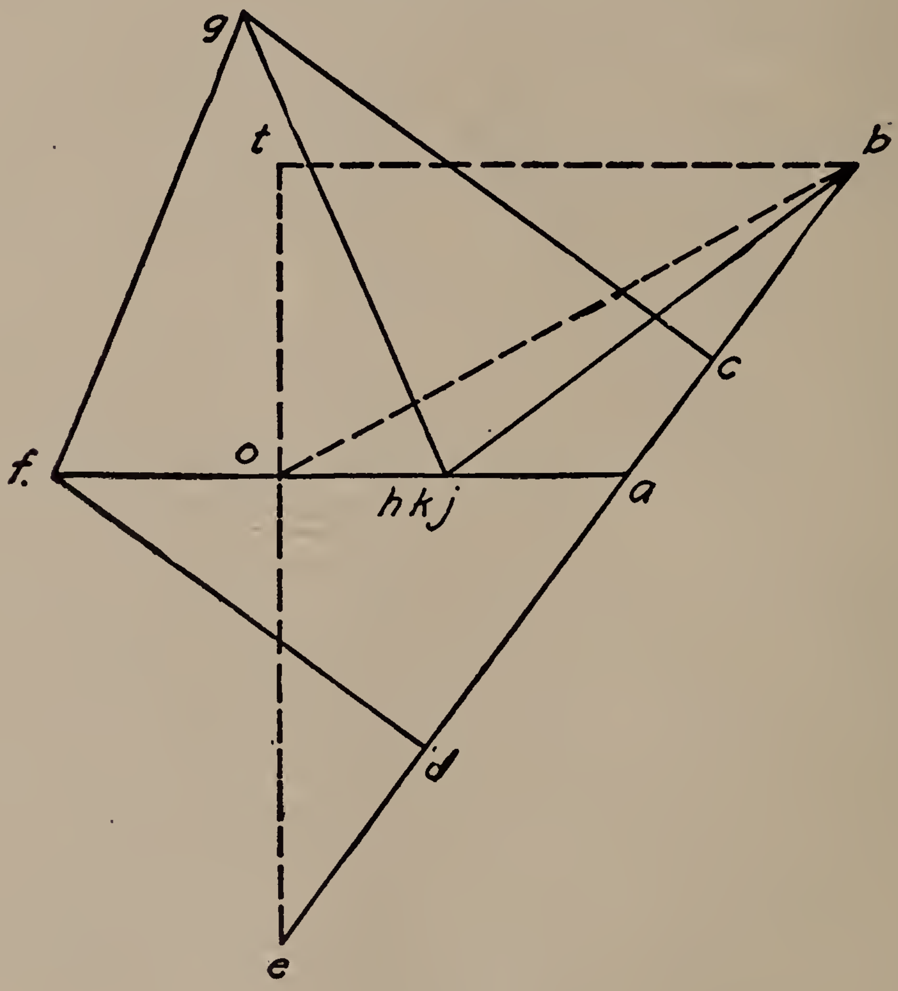


Fig. 42*b*

Figure 42*a* is a truss with the wind load distributed.

Figure 42*b* is the stress diagram for this wind load on right side. The supporting forces are here assumed to be parallel to the resultant load. Assume a roller at the right support. From this assumption determine the vertical components of the supports. These vertical components, *eo* and *ot*, will remain unchanged when any other assumption for the supports is made. *ea* and *ab* will be the supporting forces parallel to the resultant load.

Note that there is no stress in *KJ* and *JH*.

The stresses in the members for the wind load on left side will be the reverse of those shown in Fig. 42*b* for this truss. The snow load will be a vertical load and the stresses caused by it can be obtained from the dead load diagram by proportion.

The stresses for each member for each separate kind of loading can be listed in a table, and the members should be designed to resist the stresses for any possible combination of loads.

Problem: (Fig. 43.) Dead load 8000 pounds uniformly distributed over the upper chord. Wind load 4000 pounds on left side. Assume horizontal components of supports for wind load to be equal.

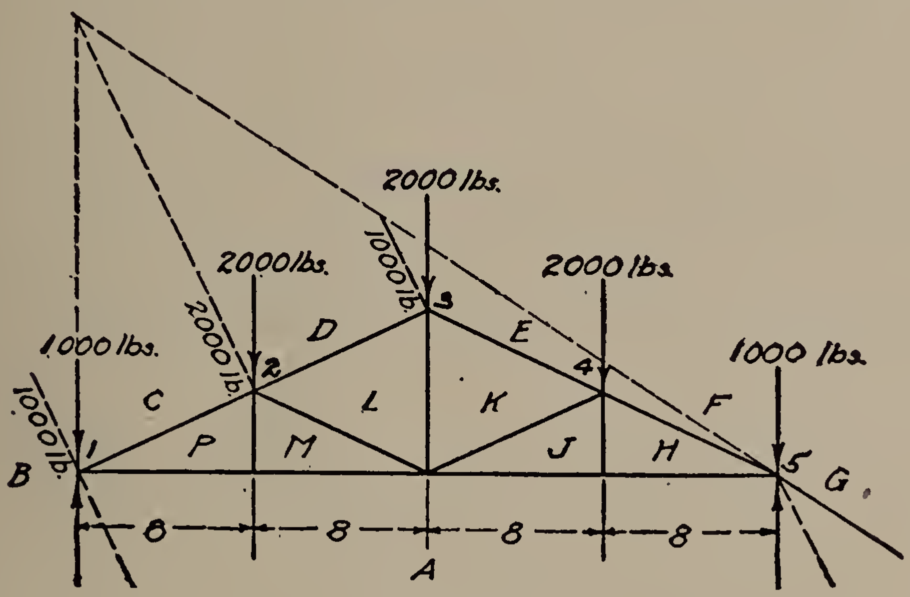


FIG. 43.

First construct the dead load diagram (Fig. 44*a*). Consider the dead load to be concentrated at the joints of the upper chord. The loads will be 1000 pounds each at joints 1 and 5 and 2000 pounds each at the other joints.

Fig. 44*a* shows the stress diagram for the dead loads.

Figure 44*b* shows the stress diagram for the wind coming on left side of the truss of Fig. 43. First assume a roller at left end of truss. This construction to determine the supporting forces with the roller is shown dotted in Figs. 43 and 44*b*.

It should be noted that all of the letters on the truss (Fig. 43) are used in the wind load diagram. As there are no loads on the right side of the truss, the wind load at the top is designated *d*(*efg*) and the right hand support is designated (*efg*)*a*.

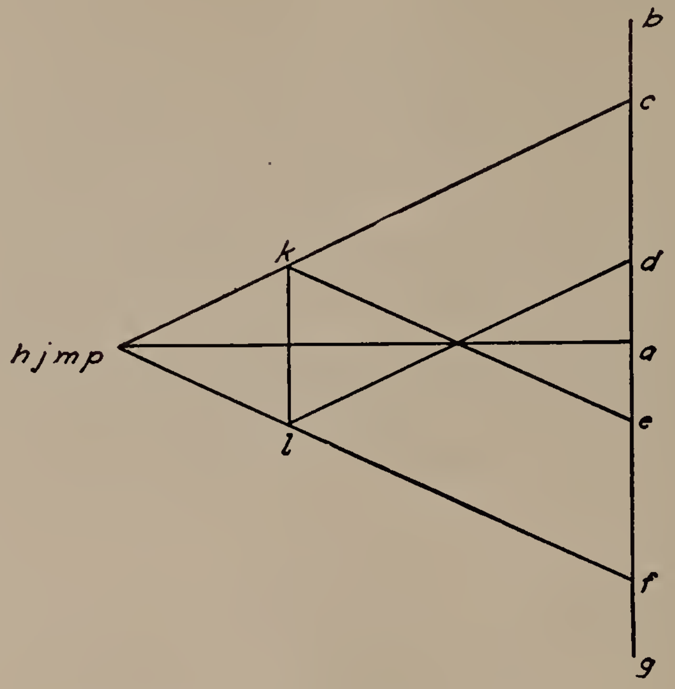


FIG. 44*a*

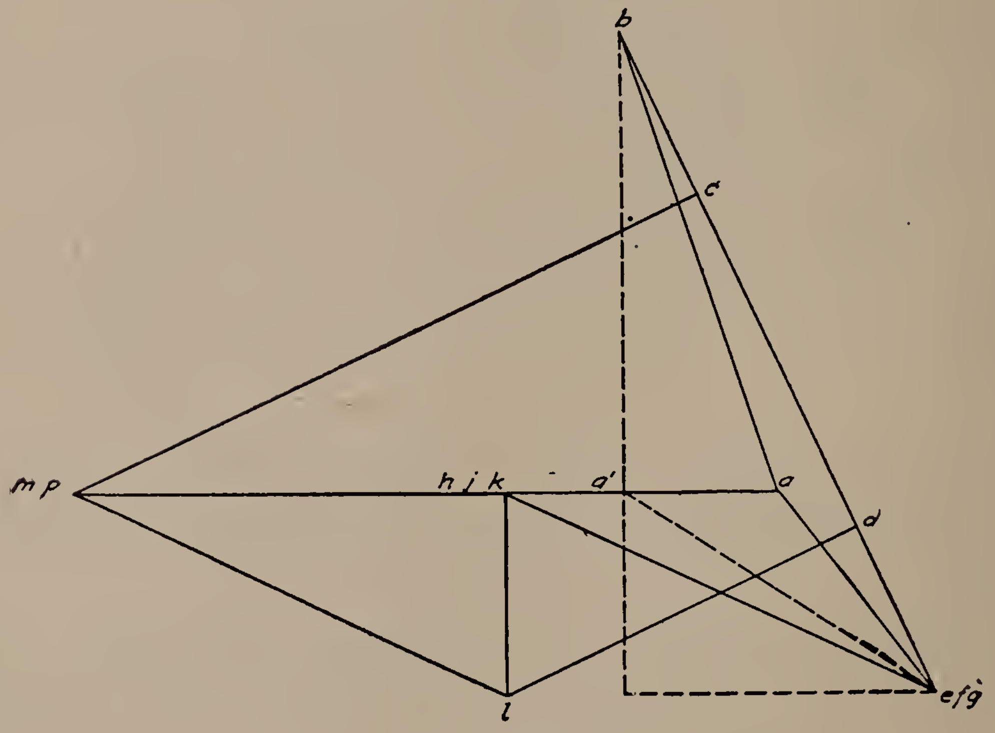


FIG. 44*b*

The truss in Fig. 45 cannot conveniently be solved graphically because joints with more than two unknown forces will be encountered. Joints 1, 2, 3, 4, 5 and 6 can be used without difficulty, but joints 7, 8, 9 and 10 will bring the graphical method to a halt. Solve for the stress in member 8–9 by passing the plane *xx* through the truss and using the method of sections $\Sigma M = 0$

with *d* as a moment axis. Place the value of this stress in the stress diagram and proceed.

Trusses are usually figured for dead loads, snow loads, wind loads and live loads if any gallery or floors are hung from the trusses themselves. The loads including the weight of the truss are considered to be concentrated at the joints.

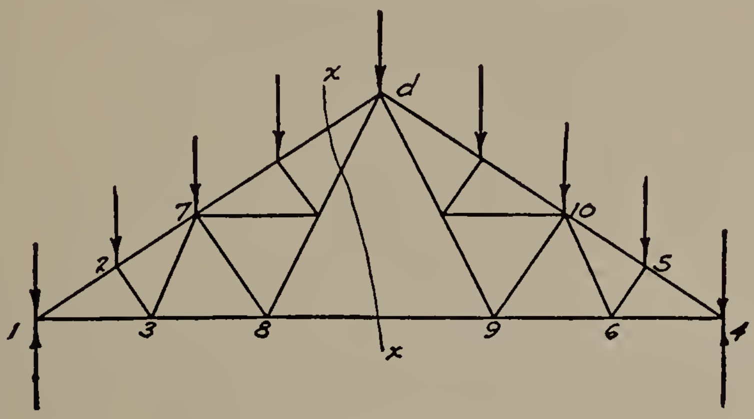


FIG. 45.

Wooden trusses are designed to contain as many compression members as possible. Steel trusses are usually made up of as many tension members as possible.

In Fig. 46, *bcegj* is the upper chord. *adfhk* is the lower chord.

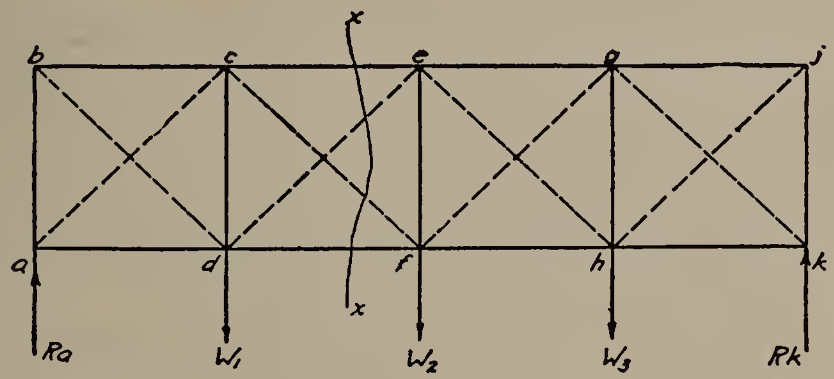


FIG. 46.

The rectangular parts, *abcd*, *dcef*, etc., are panels.

b–c–e–g–j are upper panel points. *a–d–f–h–k* are lower panel points. *ab–cd–ef–gh–jk* are verticals.

The members connecting panel points (shown dotted) are diagonals. To determine tension or compression diagonals, use the method of sections, e.g., section line *xx*.

$\Sigma V = 0$ for the forces acting upon the part of the truss on either side of *xx*, therefore the diagonal must be used that will prevent movement of one part of the truss by the other, by pulling if tension is wished, or pushing if compression is desired.

In the panel cut by *xx,* the tension diagonal will be *cf.* The compression diagonal *ed.*

If different combinations of load tend to place a diagonal in both tension and compression, two diagonals are generally used both of which are either tension or compression members. Only one of these members will be stressed at the same time.

12. Forces Whose Lines of Action are in Different Planes. *When the lines of action of the forces all pass through the same point,* only one method is of any value. This is the method of resolution. Choose three coördinate axes at right angles to each other with the origin at the point of intersection of the forces. (Fig. 47.)

Resolve each of the forces into components along OX, OY and OZ and add each set of components algebraically.

Assume that the algebraic sum of the components of the forces along OX is Oa. The arrow shows the assumed direction of this algebraic sum, Ob is the algebraic sum of the components along OY (direction assumed), and Oc is the algebraic sum of the components along OZ (direction assumed). The resultant of the system will be $R = \sqrt{(Oa)^2 + (Ob)^2 + (Oc)^2}$. The cosine of the angle that this resultant makes with the axis of X is $Oa \div R$, with the axis of Y, $Ob \div R$ and with the axis of X, $Oc \div R$. Oa is usually designated $\Sigma F \cos \alpha$ where α is the angle that any force makes with the axis of X.

$Ob = \Sigma F \cos \beta$ where β is the angle that any force makes with OY.
$Oc = \Sigma F \cos \gamma$ where γ is the angle that any force makes with OZ.

Let α_r, β_r and γ_r be the angles that the resultant makes with the axis of X, Y and Z, then

$$\cos \alpha_r = \frac{\Sigma F \cos}{R}, \qquad \cos \beta_r = \frac{\Sigma F \cos \beta}{R}, \qquad \cos \gamma_r = \frac{\Sigma F \cos \gamma}{R}$$

Conditions for Equilibrium. If the forces are in equilibrium, their resultant must be zero. Therefore, $\Sigma F \cos \alpha$, $\Sigma F \cos \beta$ and $\Sigma F \cos \gamma$ must each be equal to zero.

Determination of Unknown Forces. When the system of forces is in equilibrium and nothing is known about the unknown forces except the point of application, only one unknown force can be determined. This one unknown force must be equal and opposite to the resultant of the known forces. When the lines of action of the unknown forces are known, three unknown forces can be found if the forces are not in the same plane. More than three forces form a statically indeterminate case even if the lines of action are known.

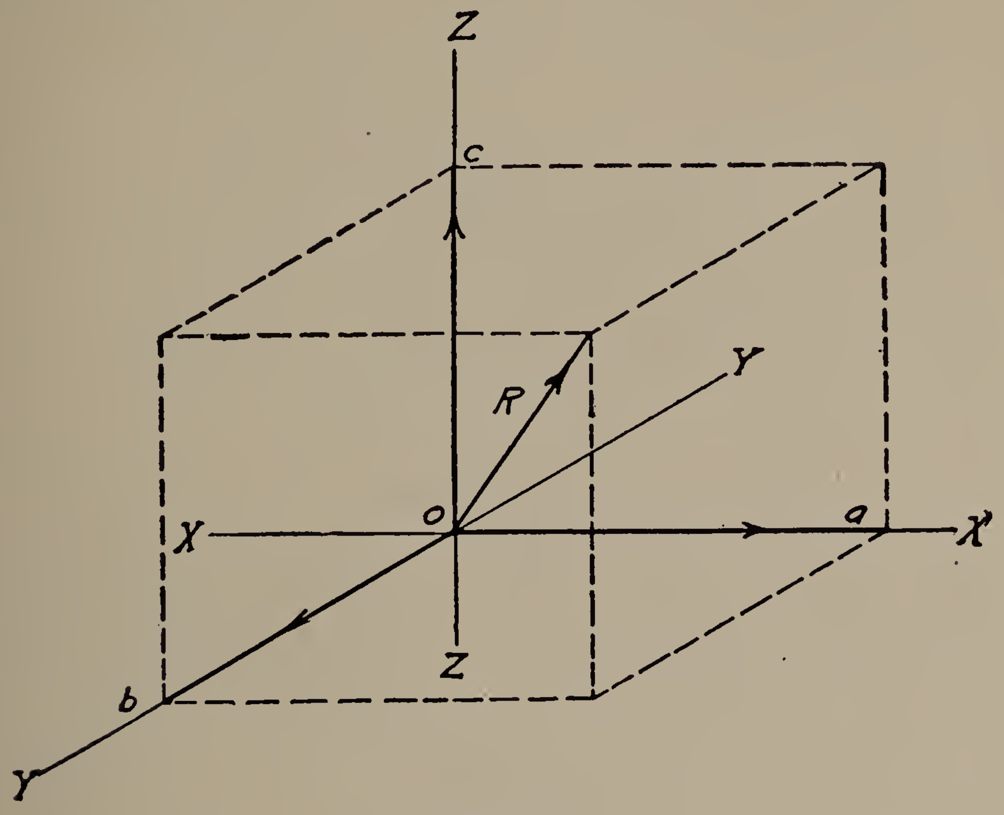


FIG. 47.

Assume directions for the unknown forces. Resolve each of the forces acting at the point into components along the coördinate axes. Add each set of components algebraically and place equal to zero. The three resulting independent equations will allow the solution for three unknown quantities. Always choose axes that will be the most convenient for the problems at hand. If the value for any of the unknown forces comes out negative, the assumed direction was wrong.

Many of the problems met with in structures coming under this case can be solved more conveniently as follows:

AB and *AC* (Fig. 48) are two members pin connected at *A*. *AD* is a guy rope also fastened at *A*. *W* is the weight to be supported from point *A*. The general method would be to consider the four forces acting at point *A*. Three of these are unknown except for line of action. The system is in equilibrium. Assume three coördinate axes and resolve the forces, assuming directions for the unknown forces.

Instead of using this general method, replace the members *AB* and *AC* by a member *AO*. *AO* is in the intersection of the plane containing *AB* and *AC*, and the plane containing *AD* and *W*. The problem now becomes two problems where the forces act in the same plane and at the same point.

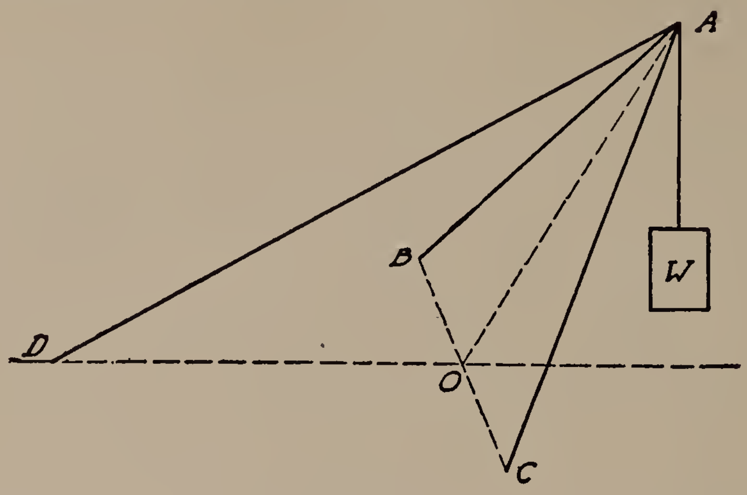


Fig. 48.

The first problem deals with the forces acting at *A* (Fig. 49). These forces are *W*, the stress in *DA* and the stress in *AO* which has replaced the members *AB* and *AC* (Fig. 48). *AO* is in the same plane as *DA* and *W*. The stresses in *DA* and *OA* may be found by either of the three methods available for the case of forces at same point and in the same plane. After the determination of stress in *OA*, the solution of another problem of the same kind will determine the stresses in *AB* and *AC*.

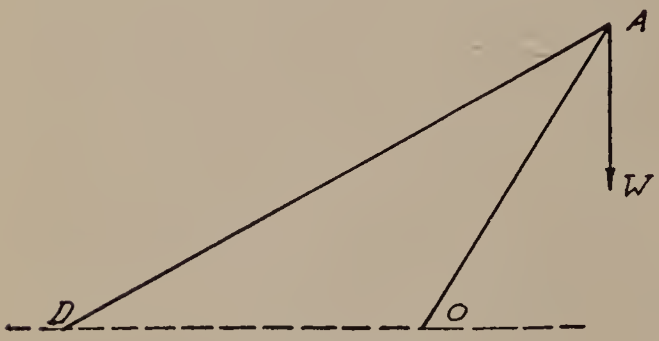


Fig. 49.

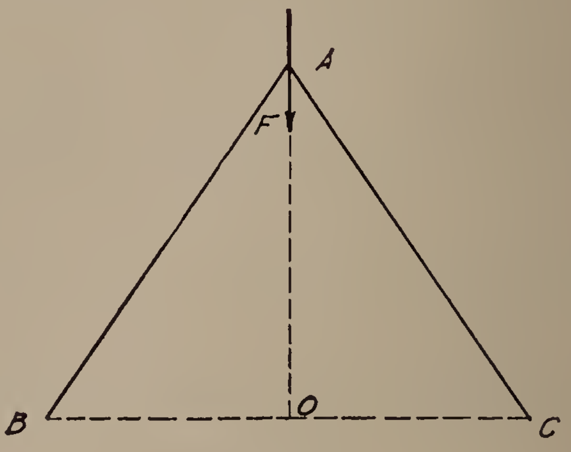


Fig. 50.

Consider the force *F* as acting at point *A*, (Fig. 50). This force *F* is equal and opposite to the stress that would be in the assumed member *AO*, Fig. 49. The member *AO*, (Fig. 49) does not exist; therefore, this force *F*, (Fig. 50) must be balanced by *AB* and *AC* for which *AO* was substituted. *AB* and *AC*, (Fig. 50) can be determined by imposing the condition of equilibrium upon the forces acting in the same plane at point *A*.

13. Parallel Forces in Different Planes. An additional couple proposition must be used in the discussion of this case.

A couple may be moved into any parallel plane provided that its moment is not changed either in magnitude or sign.

Problem: Find the resultant of the parallel forces (Fig. 51).

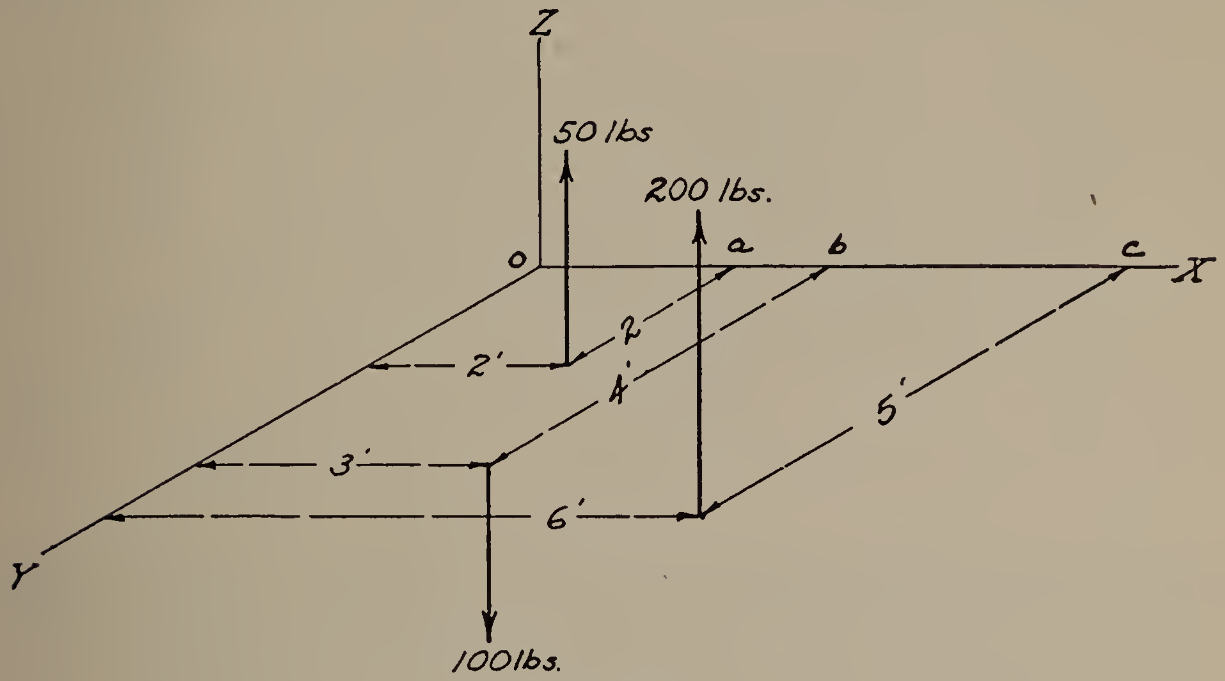


FIG. 51.

Assume a convenient origin, O, and three coördinate axes, one of which is parallel to the forces.

Replace each force by a single force acting on OX and a couple. The single forces will act at a, b and c. The couples will be in planes perpendicular to axis OX and can be moved into the same plane and combined into a single couple. The resultant of the single forces along OX, at a, b and c can be found by using the method discussed for parallel forces in the same plane. The resultant of these forces will be a force of 150 acting up. Its point of application will be at a distance from O equal to the algebraic sum of the moments of the forces about O divided by the algebraic sum of the forces (Fig. 52).

$R_1 = 50 + 200 - 100 = 150$ pounds acting up.

$\Sigma M = -50 \times 2 + 100 \times 3 - 200 \times 6 = -1000$ foot-pounds (left-handed).

Therefore $x_r = \dfrac{1000}{150} = 6\frac{2}{3}$ feet to the right of O.

The resultant of these forces is a force of 150 acting up at a distance of $6\frac{2}{3}$ feet to the right of O. (Shown in Fig. 52, dotted.)

The moments of the couples may be determined by looking along OX from the right towards the origin O. The couple used to replace the 50 pound force will be right-handed and its moment will be 100 foot-pounds. That to replace the 100 pound force will be left-handed and its moment will be 400 foot-pounds. The moment of the couple used with the 200 pound force will be right-handed and equal to 1000 foot-pounds. These couples can be combined into a single couple in any plane perpendicular to OX. The magnitude of the moment of the resultant couple will be $+ 100 - 400 + 1000 = + 700$ foot-pounds (right-handed). This couple can be moved into the plane containing the force R_1 (Fig. 52). The resultant of the combination of this couple and force in the same plane will be a single force of 150 pounds acting up at a distance from OX equal to the moment of the couple divided by the force $\frac{700}{150} = 4\frac{2}{3}$ feet. Looking towards O from the right, along OX, the moment of the couple was right-handed, therefore, the moment of the final resultant must also be right-handed looking along OX towards O. An upward force must act on the left of OX if it is to have a right-handed moment about the axis looking towards O from the right. The resultant of the forces in Fig. 51 is a force of 150 pounds acting as shown in Fig. 53.

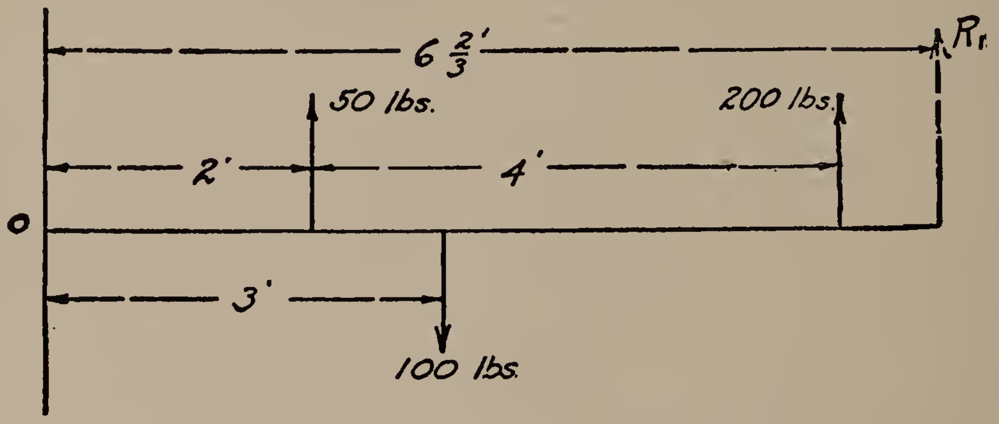


FIG. 52.

It should be noted that the original forces might have been displaced by forces acting along OY and couples in planes perpendicular to the axis OY. If this method had been used, the distance $4\frac{2}{3}$ feet from OX would have been found first and then the distance $6\frac{2}{3}$ feet from OY.

Conditions for Equilibrium. The conditions of equilibrium are: The algebraic sum of the forces must be zero, and the algebraic sum of the moments of the forces about any two axes at right angles to each other in a plane perpendicular to the forces must be zero.

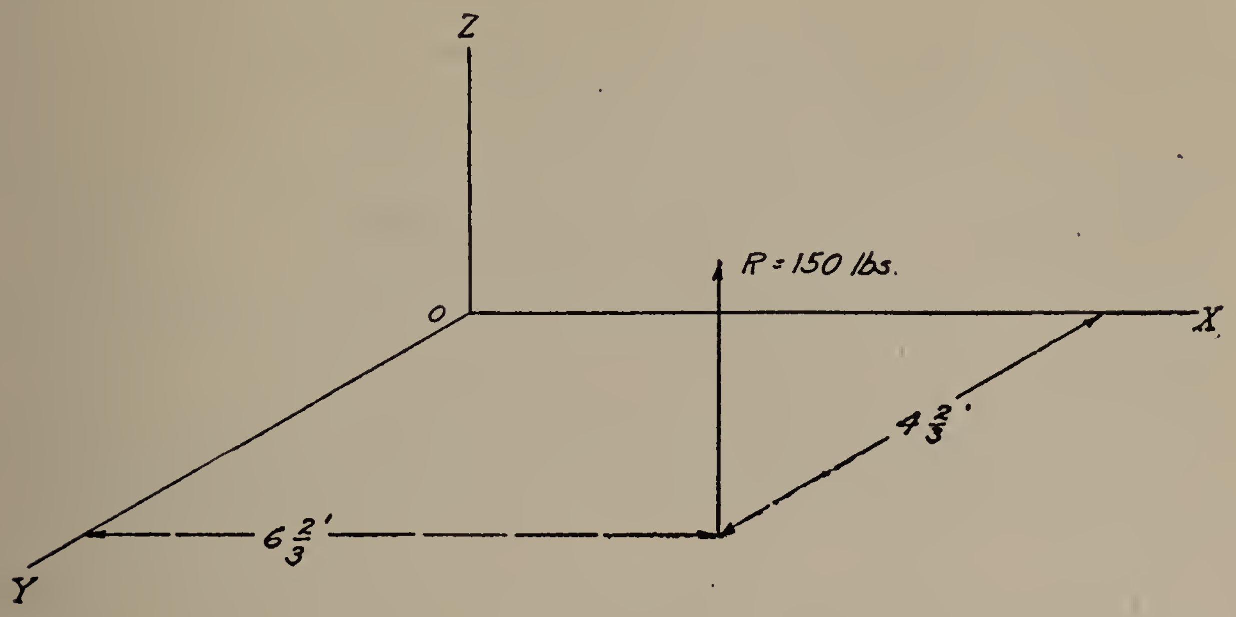


Fig. 53.

Determination of Unknown Forces. If nothing about the unknown forces of a system in equilibrium is known, only one unknown force can be determined. If the lines of action of the unknown forces are known, three unknown forces can be determined if the forces are not in the same plane.

Problem: Let the round table shown in Fig. 54 be loaded at D by a force normal to its surface and supported at A, B and C by parallel forces. The support at C can be found by taking moments about the line AB. The moment of support at C about this line must be equal and opposite to the moment of the load at D about this same line. The support at A can be obtained by taking moments about the line CB and the support at B by taking moments about AC. The values obtained can be checked by using the condition $\Sigma F = 0$.

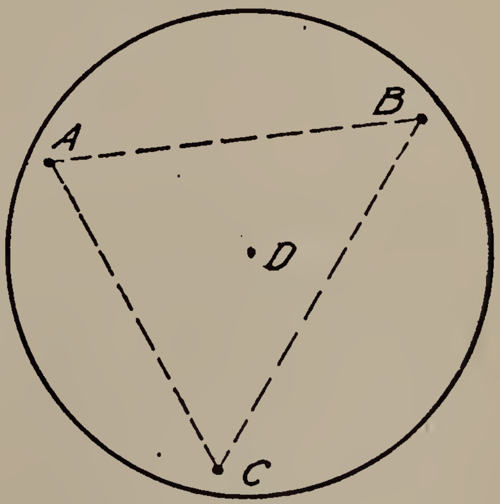


Fig. 54.

14. Couples Lying in Planes Inclined to Each Other can be combined by combining their moment axes.

The moment axis of a couple is a line whose length represents the magnitude of the moment of the couple to some scale. The arrow placed upon the line means that the couple must lie in any plane perpendicular to this line and if looked at as indicated by the arrow will have right-handed rotation (Fig. 55).

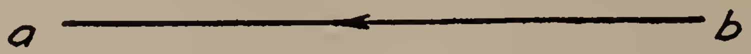


FIG. 55.

Magnitude of moment of the couple is *ab*.

Couple lies in any plane perpendicular to *ab*.

Looking in the direction of the arrow, the couple has right-handed rotation.

When a couple is said to be in the *X* plane, it is in a plane perpendicular to the axis of *X*, in the *Y* plane perpendicular to the axis of *Y*.

Given two couples, one in the *X* plane and one in the *Y* plane, find the resultant couple (Fig. 56).

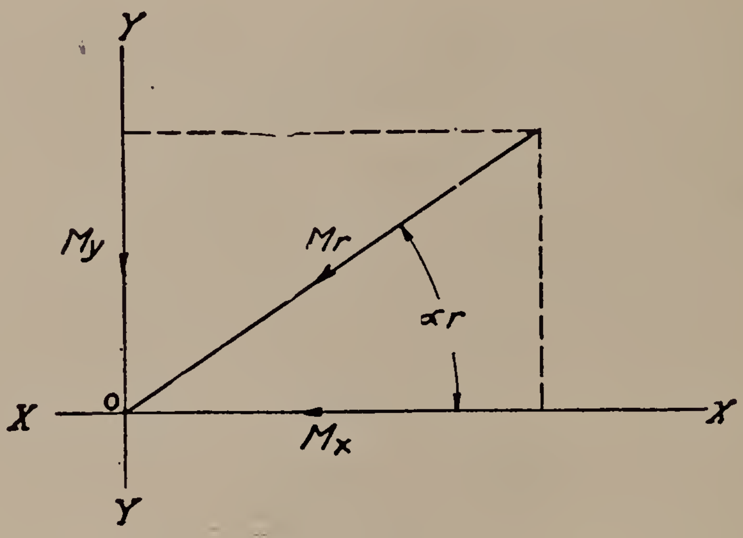


FIG. 56.

Let *OX* and *OY* be the coördinate axes and *O* the origin. On *OX* lay off from the origin a distance equal to the magnitude of the couple in the *X* plane. Lay this distance off in such a manner that looking towards the origin, the couple shall show right-handed rotation. If the distance is laid off as shown, the couple is right-handed looking towards the origin as shown by the arrow. If the couple was left-handed, looking as shown, its moment axis would have been laid off to the left of *O*. Lay off the moment of the couple in the *Y* plane in the same manner. The moment of the resultant couple will be

$$M_r = \sqrt{(M_x)^2 + (M_y)^2}, \qquad \cos \alpha_r = M_x \div M_r$$

The resultant couple has a moment = M_r and lies in a plane perpendicular to M_r and is right-handed looking towards the origin in the direction of the arrow.

15. General Case. The general case of forces acting anywhere in space will not be taken up here. It is of value for the analysis and solution of comparatively complicated problems.

Conditions for Equilibrium. The conditions for equilibrium for this case are as follows:

The algebraic sum of the components of the forces along any three axes at right angles to each other must be zero, and the algebraic sum of the moments of the forces about any three axes at right angles to each other must be zero. This general case covers all of the others.

PART II

MECHANICS OF MATERIALS

1. Stresses in General. When a structural member is loaded in such a manner as to cause bending, stresses of varying intensity are produced. Before taking up the stresses caused by bending, a study of certain physical properties of materials will be made.

By the term stress is meant the total force coming on a member. The tensile stress in a rope is 6000 pounds. The compressive stress in a cast iron column is 140,000 pounds. If the unit stress, or intensity of stress is desired, the area of the cross-section of the member and the line of action of the force applied must be known.

Stresses may be tension, compression, or shear and may be of uniform or varying intensity.

2. Uniformly Distributed Stresses. A stress or force is uniformly distributed over an area if the intensity at all parts of the area is the same. If a stress is uniformly distributed over an area its resultant must pass through the center of gravity of the area.

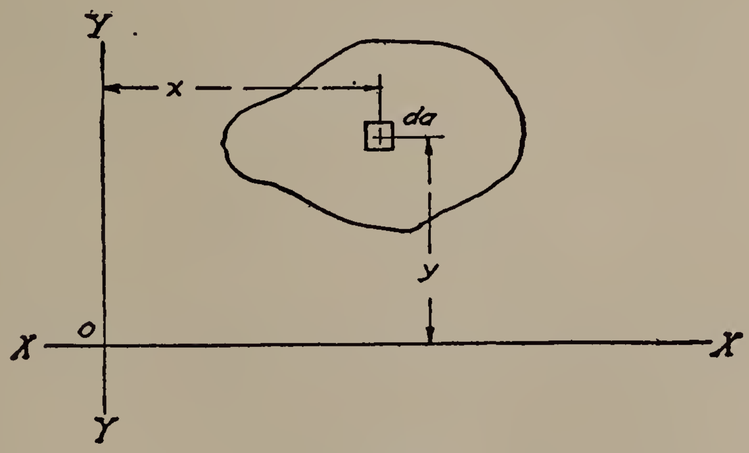


FIG. 57.

The intensity of the stress on the area (Fig. 57) is uniform and equal to p. The stress on the area is $\int\int p\,dy\,dx = pA$ where A is the area over which the stress is distributed. The point of application of the resultant of the stress can be found by taking moments of the elementary stresses about XX and YY.

Let x_r = distance from YY to the line of action of the resultant. Then

$$x_r = \int\int px\,dy\,dx \div \int\int p\,dy\,dx$$

but

$dy\,dx = dA$ and p is constant.

Therefore

$$x_r = \frac{\int x\,dA}{\int dA}.$$ This is the expression for the distance from YY to the center of gravity of the area. In the same way

$$y_r = \frac{\int y\,dA}{\int dA}.$$ Therefore, y_r = distance from XX to center of gravity of the area. Therefore, the point of application of the resultant of a stress of uniform intensity on an area is at the center of gravity of the area.

3. Uniformly Varying Stress. The only other common distribution is a uniformly varying stress. Consider the case of an area over which there is a stress that varies uniformly along an axis of symmetry of the area.

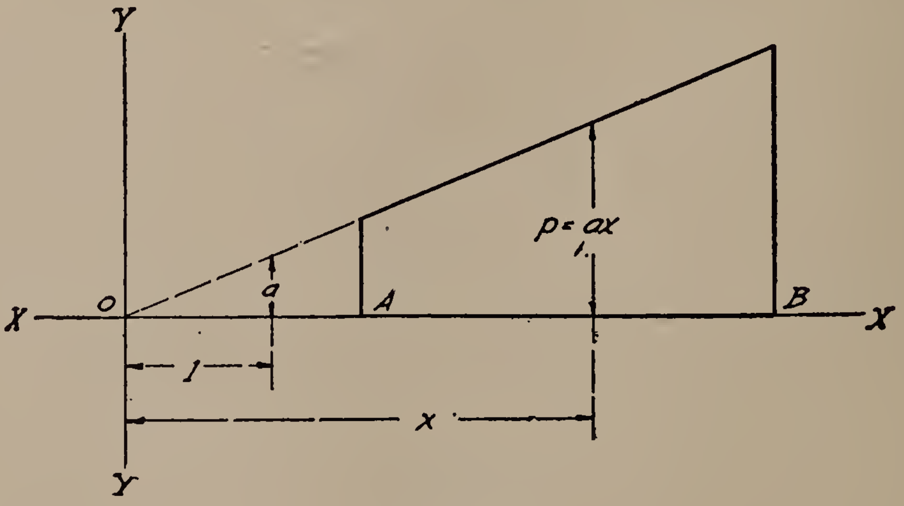


FIG. 58.

AB (Fig. 58) is the trace of such an area upon which there is a normal stress of uniformly varying intensity. If this area was extended and the intensity of the stress continued to vary as it does along AB, there would be some line on the area where the intensity of the stress would be zero. This line is the *neutral axis* of the stress, and all measurements should be made from it.

If the intensity of the stress at a distance unity from the neutral axis is a, the intensity of stress distance x from the neutral axis is ax. The resultant stress on the area is $P = \int ax\,dA = a\int x dA = ax_0A$. dA is a strip dx wide across the area at right angles to AB. Therefore the resultant of a uniformly varying stress is the area over which the stress is distributed, multiplied by the intensity of the stress on the center of gravity of the area.

The point of application of the resultant stress may be found as follows: Take moments of the elementary stresses about the neutral axis O, (Fig. 58) and divide by the resultant P.

$$x_r = \frac{\int ax^2\,dA}{ax_0A} = \frac{I}{x_0A}$$

where I is the moment of inertia of the area with reference to the neutral axis, and x_0A is the moment of the area about the neutral axis. The point of application of the resultant stress is on the axis of symmetry at a distance x_r from the neutral axis.

$I = I_0 + x_0^2A$ where I_0 is the moment of inertia of the area with reference to an axis passing through the center of gravity of the area and parallel to the neutral axis.

Therefore, $$x_r = \frac{I_0 + x_0^2A}{x_0A} = \frac{I_0}{x_0A} + x_0.$$

Therefore, $$(x_r - x_0) = \frac{I_0}{x_0A}$$

$(x_r - x_0)$ is a distance between the point of application of the resultant stress and the center of gravity of the area over which the stress is distributed. Water pressure is an example of uniformly varying stress. If the neutral axis of a stress is in the area, the direction of the stress reverses.

When the neutral axis of the stress is at the center of gravity of the area, the resultant of the uniformly varying stress is a couple. When the stress varies uniformly along some other axis than an axis of symmetry, or when the area does not have such an axis, two axes of reference must be chosen and a second coördinate determined for the line of action of the resultant.

4. Unit Stresses, Strains and Modulus of Elasticity. A stress that is normal to the area upon which it acts is a tension or compression. When the stress is parallel to the area, it is known as a shearing stress.

No material is actually rigid although if properly designed members of structures are practically so. Any stress upon a body will cause a distortion, known as a strain. Strains are distortions per unit of length caused by stresses.

A tensile strain is the elongation of a body divided by the original dimension of the body in which the elongation was measured. A compressive strain is the compression in a body per unit of length.

A shearing strain is shown by Fig. 59.

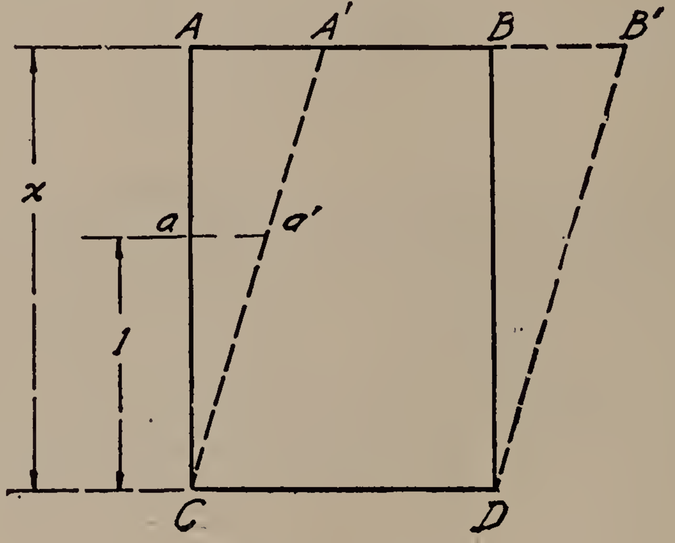


FIG. 59.

Under the action of a shearing stress, the plane AB slides by plane CD into the dotted position $A'B'$. The amount of distortion is AA' and this takes place in length x.

If this distortion AA' is divided by x, the distortion aa' in a unit of length is the shearing strain.

When a material is distorted by a stress, the relation between the unit stress and the unit distortion in the direction of the stress is known as the Modulus of Elasticity (E).

Let p_t equal the unit stress in tension and p_c equal the unit stress in compression.

Let $e_t =$ the elongation in a unit of length, $e_c =$ the compression in a unit of length.

Then $E_{(\text{tension})} = \dfrac{p_t}{e_t}$ and $E_{(\text{compression})} = \dfrac{p_c}{e_c}$.

Stresses that are not normal to the area upon which they act may be resolved into components normal and parallel to the area. The normal components are tensions or compressions and the parallel components are shears.

Let the rod (Fig. 60) be acted upon by the forces P that act along the axis of the rod. The stress at area AA will be all normal and the intensity will be P divided by area of AA. The stress on the area BB is not normal and may be resolved into the normal stress P_n and the shearing stress P_s. The intensity of tension on this area is uniform and equal to P_n divided by the area of BB. The intensity of shearing stress on this area is uniform and equal to P_s divided by area of BB. The maximum intensity of shear is on a plane making 45° with the normal plane AA, and is equal to one half of P divided by area AA. In order that the intensity of stress at any point in a stressed body may be found, the plane upon which the point is located must be determined.

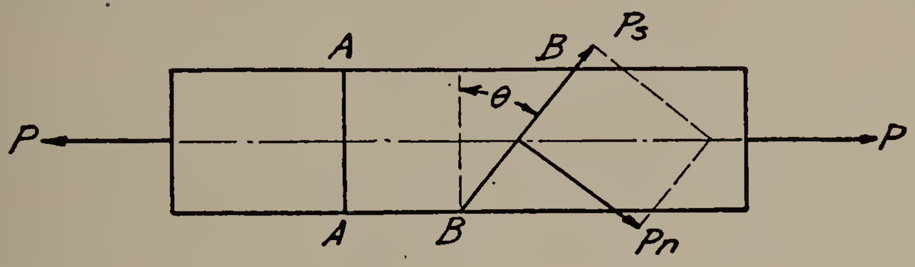


FIG. 60.

5. Stress-Strain Diagram. When a piece of ductile steel is tested in tension a stress-strain diagram may be plotted (Fig. 61). Ordinates are unit stresses and abscissae are unit distortions or strains. The specimen is held in the jaws of a testing machine and the load applied slowly. Readings are made by means of a suitable measuring instrument to determine the elongation at various loads. A definite length is taken in which the elongation is measured. As the unit stress slowly increases, it will be noted that the strain increases proportionately up to a unit stress of p_1. At this unit stress, the strain ceases to be proportional and begins to increase slightly faster than the stress. This intensity of stress p_1 is known as the proportional limit of the material. If there is no permanent distortion of the material under the unit stress p_1 this unit stress is the elastic limit of the material. The proportional limit and elastic limit are the same for steel, but not for all materials. The permanent distortion caused by any stress is known as permanent set. *The elastic limit of a material is the largest unit stress that can be applied without causing a permanent distortion or set.*

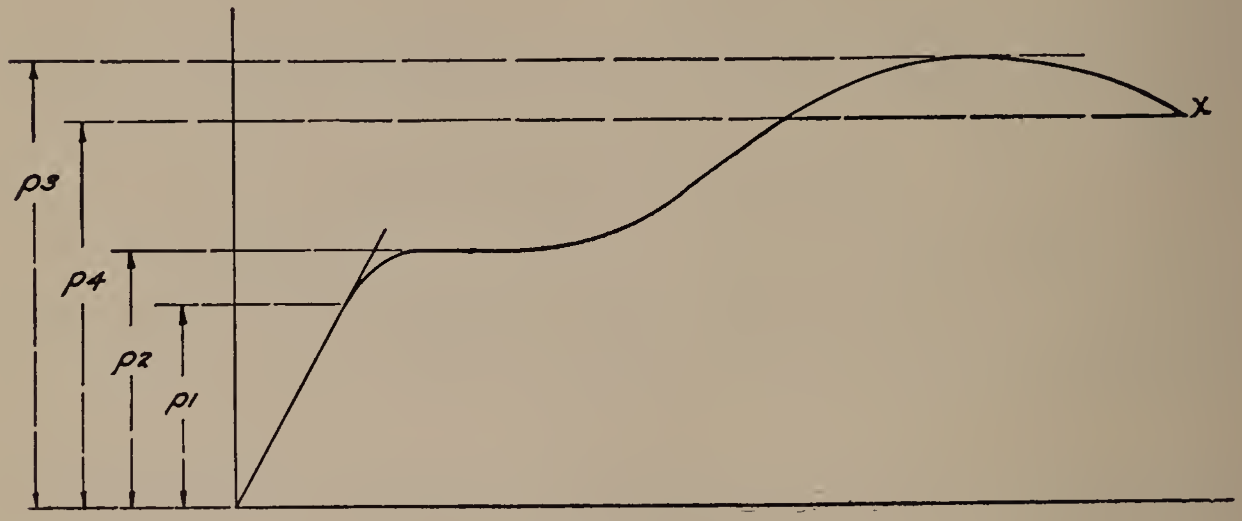


Fig. 61.

The Modulus of Elasticity of a material is a constant up to the elastic limit. Hooke's Law states that the unit stress is proportional to the strain and, therefore, this law holds up to the elastic limit of the material.

As the unit stress increases beyond p_1, the rate of change of the strain to unit stress grows larger and at the unit stress p_2 becomes rapid enough to be observed without the aid of the delicate measuring instrument required to determine the elastic limit. This unit stress p_2 is known as the yield point of the material. The terms yield point and elastic limit are very commonly confused, and the term elastic limit is often used when yield point is meant. Great care should be exercised to use the proper term. Yield point is usually determined when testing steel because it is much more easy to obtain than the true elastic limit. The relation between yield point and elastic limit of structural materials is known.

Continue the loading of the specimen beyond the yield point p_2. As the load increases, the distortion increases much more rapidly than before. When the unit stress reaches p_3, a maximum is determined and local contraction of the specimen begins. This local action depends upon the ductility of the material. The specimen breaks at a lower load than the maximum in ductile materials on account of the local reduction of area and the comparatively slow application of load.

It is customary to determine unit stresses by dividing the load at any time by the original area of the cross-section of the specimen. This accounts for the apparent fall of the unit stress to p_4 at x. As a matter of fact, if the load at breaking had been divided by the reduced area at that instant, a unit stress considerably larger than that shown at p_4 would have been obtained. Elongations in specified lengths and contraction of area are measures of ductility. It should be noted that the percentage elongation in different lengths of ductile specimens is not the same due to the fact that the local contraction takes place in a comparatively short distance. A stress strain plot for a compression test can be obtained in the same manner.

The merit number of a material is the breaking tensile strength in pounds per square inch multiplied by the elongation per cent of 2 inches. The values are obtained using a specimen about $\frac{1}{2}$ inch in diameter.

6. Important Qualities of Materials.

Elastic Limit.
Yield Point.
Modulus of Elasticity.
Maximum or Breaking Strength.
Elongation per cent of some given length.
Reduction of area per cent of original area.

The Maximum Load on a specimen is the maximum or breaking strength multiplied by the area. A safe or working strength for any material is obtained by dividing the breaking strength by what is known as a factor of safety. This value must be considerably less than the elastic limit of the material. Working Loads and Breaking Loads as defined above are considered to be gradually applied loads.

Safe working strengths for the more common structural materials can be found in Handbooks.

The Modulus of Elasticity of structural steel of any grade is the same. About 30,000,000 pounds per square inch. The modulus of elasticity for compression is practically the same as for tension in the case of steel.

Problem: Find the elongation of a steel rod 10 feet long and 2 square inches in cross-section under a load that will cause a working intensity of stress of 15,000 pounds per square inch.

$$E = \frac{p}{e}. \quad \text{Therefore } e = \frac{p}{E} = \frac{15{,}000}{30{,}000{,}000} = \frac{1}{2000} = \text{strain}$$

Elongation is equal to strain multiplied by length, therefore,

$$\text{elongation} = \frac{1 \times 10 \times 12}{2000} = .06 \text{ inch}$$

The working load is 15,000 × 2 = 30,000 pounds.

7. Temperature Stresses. The coefficient of linear expansion of a material is the amount a unit length will change its dimension for one degree change of temperature.

If a rod is fastened rigidly at the ends and the temperature varies, the change in stress intensity will be the same as would be caused by a stress that would distort the rod the same amount as it would distort if one end was free when the temperature changed.

Problem: A steel rod is fastened to rigid supports. The temperature falls 100 degrees. Find the tension in the rod.

$$E = 30{,}000{,}000 \text{ pounds square inch.} \quad \text{Coefficient} = .0000065$$

$$.0000065 \times 100 = .00065 = \text{strain}$$

$$p = Ee = .00065 \times 30{,}000{,}000 = 19{,}500 \text{ pounds square inch}$$

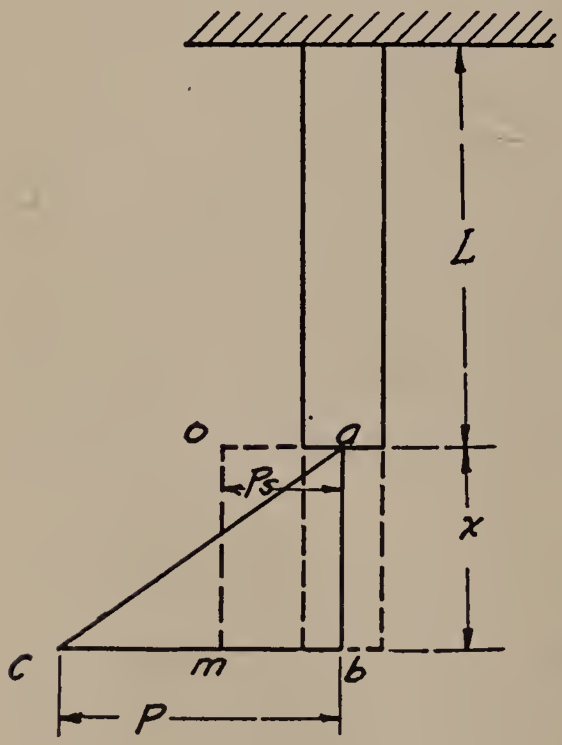


FIG. 62.

8. Suddenly Applied Loads. A study of the effect of a suddenly applied load brings in the term resilience. Resilience is the amount of work that can be obtained from a strained body. If the body is not stressed beyond the elastic limit of the material, the amount of work that can be got back is the same as that put in. If the material is stressed beyond its elastic limit some of the work put in is used in permanently distorting the body.

The rod (Fig. 62) is acted upon by a gradually applied force P where P is the working load. The elongation caused by the load is the working elongation x. The work done in elongating the rod this amount is the working resilience.

The working resilience is equal to the working load multiplied by the working elongation and divided by two. If x is measured in inches and P in pounds, the work done will be inch pounds and will be represented by the triangle *abc*. Under no circumstances can the rod be elongated more than the working elongation x. If a sudden load P_s is applied, the work done will be represented by the rectangle *aomb* the area of which is P_s multiplied by x. The work done in this case must be the same as before.

Therefore,
$$P_s = \frac{P}{2}$$

The safe suddenly applied load that a rod can withstand is one-half of the safe gradually applied load.

9. Shearing Stress. A common example of shearing stress is met in riveted joints.

The rivets in Fig. 63*a* and Fig. 63*b* prevent the plates from pulling apart. In order to carry out this function successfully, they must resist a tendency to shear. This shearing force is assumed to be uniformly distributed over the cross-section of the rivets. The intensity of the shearing stress in Fig. 63*a* is P divided by the area of cross-section of the rivet. In Fig. 63*b* the intensity of the shearing stress is one-half P divided by the area. The rivet in Fig. 63*a* is in single shear and the rivet in Fig. 63*b* is in double shear. The analysis of riveted joints is a part of "Structural Design," but a brief discussion will not be out of place here.

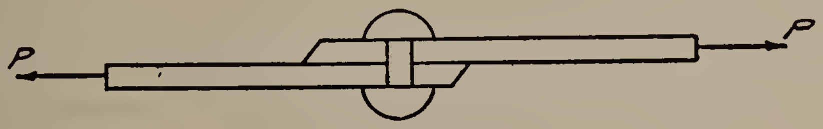


Fig. 63*a*.

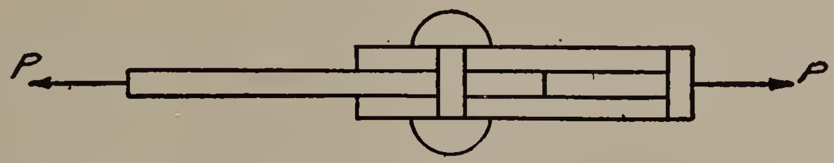


Fig. 63*b*.

The single riveted lap joint (Fig. 64) is a tension joint. Thickness of plates is t, diameter of rivets is d, distance between rivets center to center is p = pitch. Working intensity of shearing stress is f_s, of tension f_t and of compression f_c for both plates and rivets.

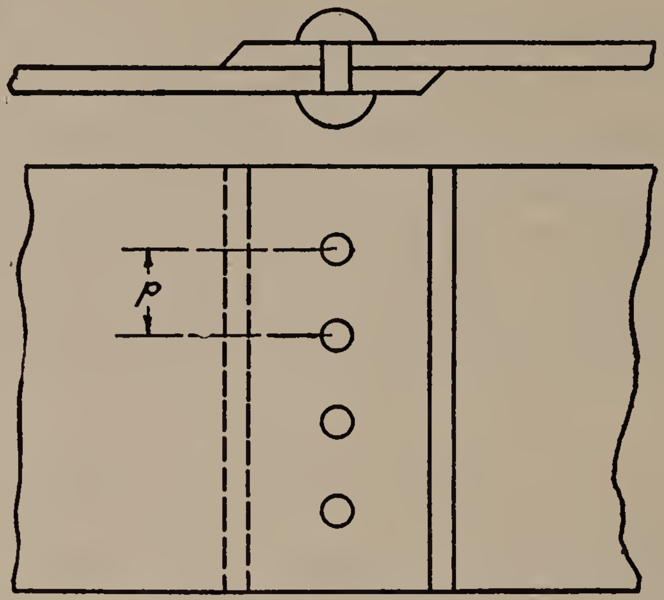


FIG. 64.

The resistance offered by the rivet to shearing is $\frac{\pi d^2}{4} f_s$. In order to bring about a shearing stress in a rivet, there must be a bearing pressure between the plates and the rivet. This bearing pressure varies as indicated (Fig. 65).

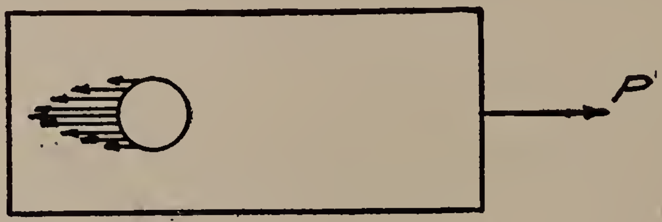


FIG. 65.

It is usually assumed that the intensity of compression between rivet and plate is uniformly distributed over the projection of the bearing area. The projection of the bearing area is the diameter of the rivet multiplied by the thickness of the plate.

The resistance offered by the rivet to compression is, therefore, tdf_c where t is the thickness of the thinnest plate bearing upon the rivet and d is the diameter of the rivet. The pitch is determined by equating the resistance offered by the part of the plate between rivet holes to tension to the smaller of the shearing or bearing resistance offered by the rivet. The resistance offered to tension is $(p - d)\, tf_t$ where p is the pitch, d is the diameter of the rivet and t is the thickness of the plate (Fig. 64). The maximum and minimum allowable pitches are determined largely by practical considerations.

A butt joint (Fig. 66) is analyzed in the same manner noting that the resistance offered by the rivet to shear is twice the area times the allowable intensity of shear.

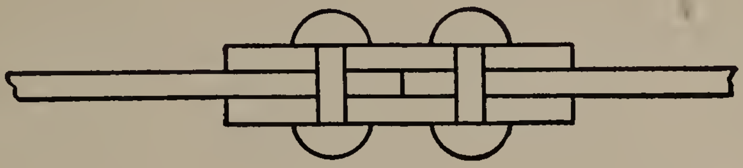


FIG. 66.

In the design of structural joints, practice dictates very largely but the general method may be stated as follows: Enough rivet sections must be provided to prevent the intensity of bearing or shear from exceeding the limits set for the particular problem.

Any discussion for the determination of dimensions for laps, spacing of rivets, etc., is out of place here.

10. Bending Stresses. The determination of bending stresses requires the derivation of certain formulae. The most common of these formulae are only good under certain rather limited conditions. In design, the conditions of any problem must be carefully analyzed and the formulae that come nearest to the case must be properly modified before being used.

The common beam formulae are suitable for use under certain limitations. The term beam will be used to designate a member subjected to bending stresses.

Limitations for Common Beam Formulae. The beam must be straight, uniform in cross-section and the cross-section must have at least one axis of symmetry. The dimensions of the cross-section must not be large in comparison with the length.

The material of the beam must be homogeneous and must not be stressed beyond its elastic limit. The Modulus of Elasticity for compression must equal that for tension.

The external forces must all lie in the same plane and must all be perpendicular to the axis of the beam. The plane containing the forces must divide the cross-section symmetrically. By cross-section is meant a section at right angles to the axis of the beam.

If there is any departure from these limitations, suitable changes must be introduced into the general formulae.

The derivation of the general formula for stresses due to bending under the stated limitations, depends upon the conditions of equilibrium for a set of forces in a single plane

$$\Sigma M = 0 \qquad \Sigma F_x = 0 \qquad \Sigma F_y = 0$$

The beam (Fig. 67) comes under the limitations of the theory. If this beam is to be serviceable, it must be acted upon by a set of balanced forces. All of the external forces are assumed to be in the same plane and are parallel, therefore, $\Sigma F = 0$ and $\Sigma M = 0$. This will allow the supporting forces R_a and R_b to be determined.

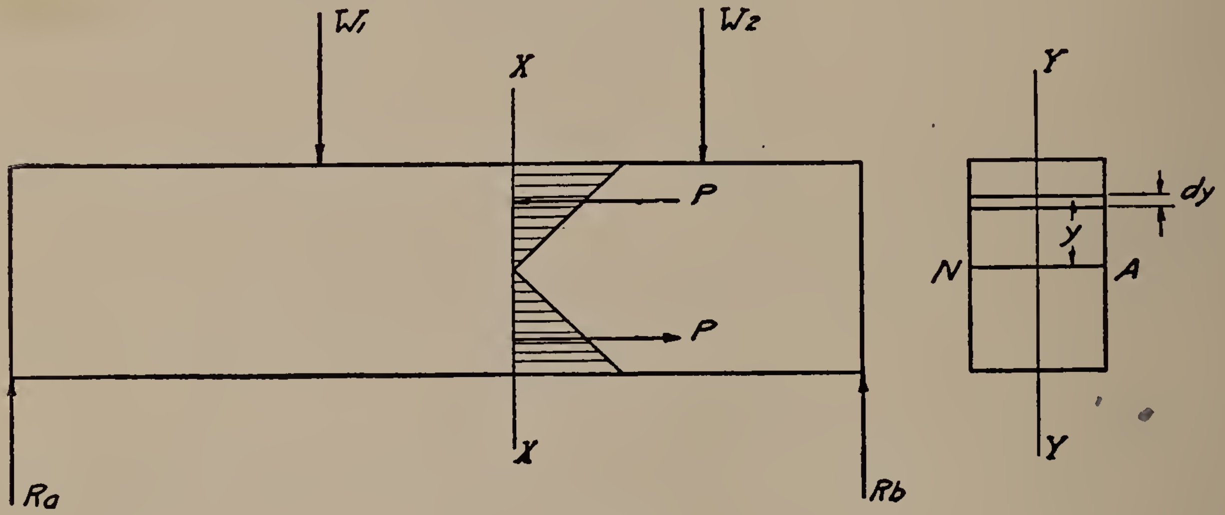


Fig. 67.

If the whole beam is in equilibrium under the action of the external forces, any part of the beam must be in equilibrium under the action of the forces acting externally to that part.

Consider the part of the beam on either side of the plane XX. XX is any plane at right angles to the axis of the beam. The forces acting externally to this part of the beam, say the part on the left of XX, are the external loads and supports and the internal stresses with which the part of the beam on the right of the section acts on the part on the left. These internal stresses now become external forces when the part of the beam on one side of the section is considered. One side of the beam will be in tension and the other in compression, therefore there must be some line across the section where the intensity of stress will be zero. Designate this line as the Neutral Axis of the stress ($N A$) The resultant of the stress acting upon any area dA of elementary thickness and parallel to NA will be where the plane YY containing the external forces cuts the area. Therefore, the resultant of all the internal stresses must lie along YY, therefore, all of the forces acting upon the part of the beam to the left of XX will be in the same plane.

The algebraic sum of the vertical components of these forces must be equal to zero, therefore, the resistance offered by the part of the beam to the right of XX to that on the left to prevent it moving vertically must be equal and opposite to the algebraic sum of the external forces acting upon the part of the beam to the left of XX.

The shearing force at any cross-section of a beam is equal in magnitude to the algebraic sum of the external forces acting upon the part of the beam on either side of the section. It should be noted that this shearing force is the total force acting parallel to the section and not intensity.

For equilibrium, the algebraic sum of the horizontal components of the forces acting upon the part of the beam to the left of XX must be zero. The external forces have no horizontal components, therefore the algebraic sum of the normal components of the forces exerted by the part of the beam to the right of XX upon the part to the left must be equal to zero.

In order to deal with these normal components, in this case horizontal components, certain assumptions must be made.

Assume that a plane cross-section remains plane after bending. From this assumption it is derived that the strains upon any fiber are proportional to the distance of the fiber from a point in the cross-section that is not strained at all. The term fiber is used to denote any part or layer of the beam.

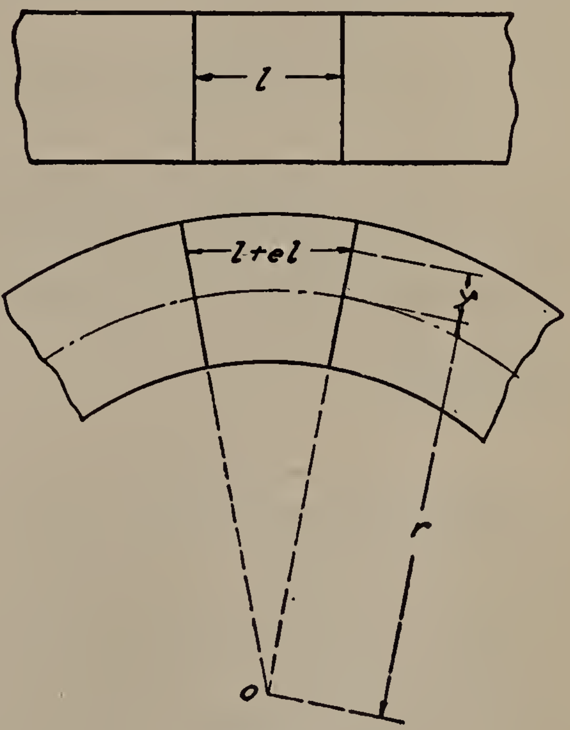


Fig. 68.

Two plane sections (Fig. 68) are taken a distance l apart. l is *very small.* After bending the planes are still plane (first assumption). Some of the fibers of the beam are elongated and some shortened. Assume that the beam bends in the arc of a circle between the two planes and that the center of curvature is O. r is the radius of curvature of the neutral layer. Let e be the strain on any fiber. Then the elongation or contraction in length l will be el and the length of a fiber distance y from the part of the beam that is not strained will be, after bending, $l + el$.

From similar triangles $(l + el) \div l = (r + y) \div r$

Therefore $$e = \frac{1}{r}y$$

This shows that the strain in any fiber of the beam is proportional to the distance of the fiber from that part of the beam that is not strained at all.

The second assumption is not really an assumption for most structural materials. It is the limitation that no fiber shall be stressed beyond the elastic limit of the material. If this is the case, the unit stresses are proportional to the strains and therefore the stress over the cross-section is a uniformly varying one if the modulus of elasticity for tension and compression are the same.

The summation of this uniformly varying stress must be zero. Therefore the resultant is a couple whose moment is aI, where a is the intensity of stress at distance unity from the neutral axis of the stress, and I is the moment of inertia of the area with respect to the neutral axis.

When the resultant of a uniformly varying stress is a couple, the neutral axis must pass through the center of gravity of the area.

$$P = ax_0 A = 0 \qquad \text{therefore, } x_0 = 0$$

a is the intensity of stress at distance unity from the neutral axis. a is, therefore, equal to the intensity of stress p, at a distance y from the neutral axis divided by y.

Therefore $$a = \frac{p}{y}$$

The moment of the couple formed by the normal stresses is aI, and therefore equal to $\frac{p}{y}I$.

ΣM = zero. Therefore, the algebraic sum of the moments of all the forces acting upon the part of the beam to the left of the section XX (Fig. 67) about any point in their plane must be zero.

Take an axis at neutral axis of the section. Then the algebraic sum of the moments of the loads and supports about this axis must be balanced by the moment of the couple formed by the tensions and compression at the section, or

$$M = \frac{p}{y} I \qquad \text{or} \qquad p = \frac{M_y}{I}$$

M is the bending moment at the section and is equal in magnitude to the algebraic sum of the moments of the external forces acting upon the part of the beam to the left of the section taken about an axis in the section.

It should be noted that in the determination of Shearing Force and Bending Moment at any section, the part of the beam on the right of the section could have been used just as well.

If the maximum fiber stress is wanted, use the formula

$p = \frac{My}{I}$ where p = maximum intensity of stress.

M = maximum bending moment for the beam.

y = distance from the neutral axis (center of gravity of cross-section) to the most strained fiber. (One farthest from neutral axis.)

I = moment of inertia of the cross-section about the neutral axis. (The neutral axis is in the plane of the cross-section, passes through its center of gravity and is perpendicular to the plane of loading.)

When working intensity of stress is to be used, designate it as f.

$$f = \frac{My}{I} = \text{working intensity of stress}$$

Care must be exercised with regard to the units. Bending moment should be expressed in inch pounds and the dimensions of the cross-section in inches.

When solving a beam problem proceed as follows:

Analyze the problem carefully to determine if the limitations of the beam theory are fulfilled.

If the limitations are fulfilled, determine the supporting forces, then plot the shearing force diagram. Find the point or points of zero shear and determine the bending moments for these sections. Take the largest of these and place it in the formula $f = \frac{My}{I}$.

11. Shearing Force and Bending Moment. Assume the convention for shearing force as shown (Fig. 69). The tendency of the right-hand part of a beam to slide down by the left will be considered positive shear.

Any force that tends to bend a beam concave upwards produces what will be considered to be a positive bending moment,

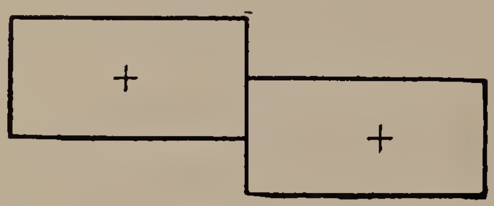


Fig. 69.

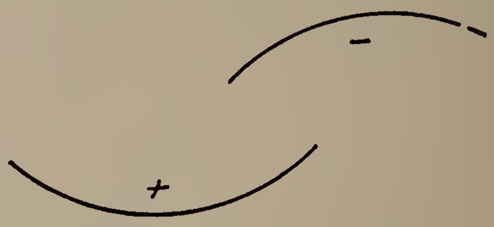


Fig. 69*a*

a force tending to bend the beam convex upwards produces a negative bending moment (Fig. 69*a*).

A beam supported at the ends and loaded with a single concentrated load has shearing force and bending moment diagrams shown by Figs. 70*a* and 70*b*.

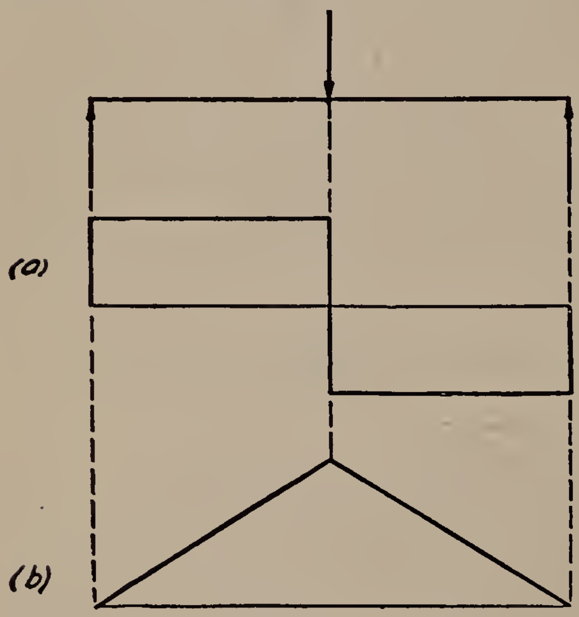


Fig. 70.

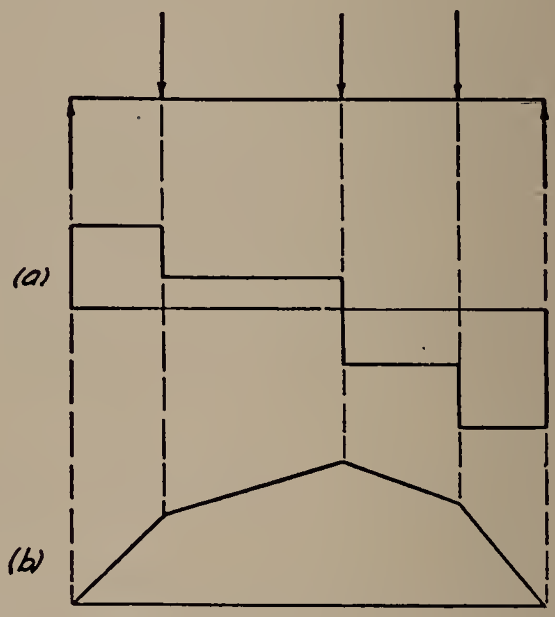


Fig. 71.

Figs. 71*a* and 71*b* show diagrams where there is more than one load. It should be noted that zero shear may be at any one of the loads, depending upon their relative magnitude and position.

A beam with a uniformly distributed load has shearing force and bending moment diagrams shown by Figs. 72*a* and 72*b*.

Figure 73*a* shows shearing force diagrams for beams having both distributed and concentrated loads. Fig. 73*b* shows the bending moment diagrams corresponding.

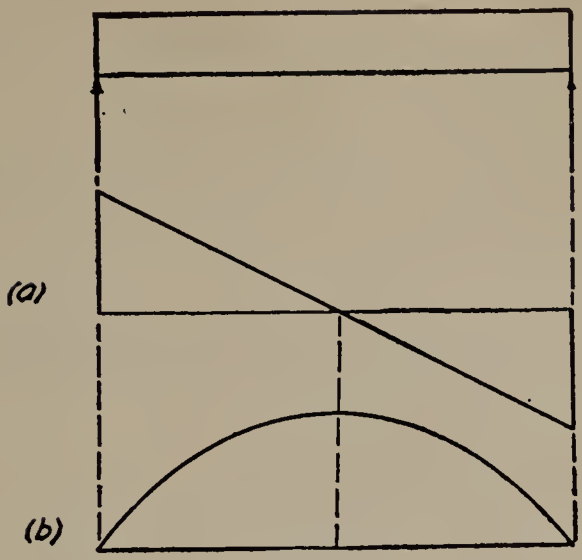


Fig. 72.

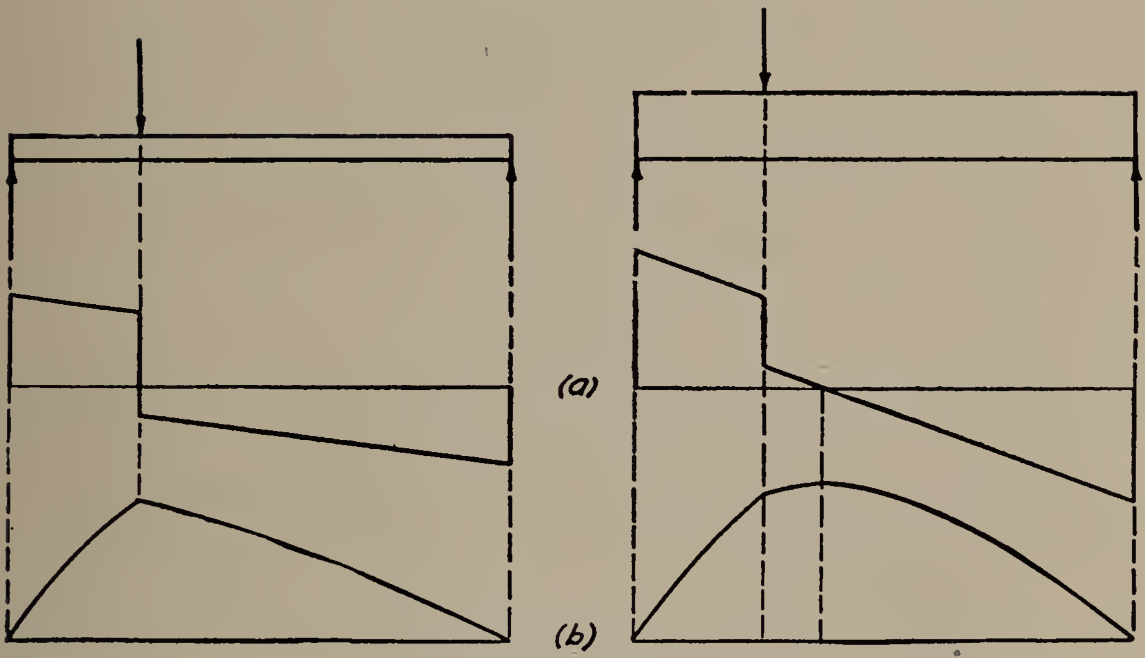


Fig. 73.

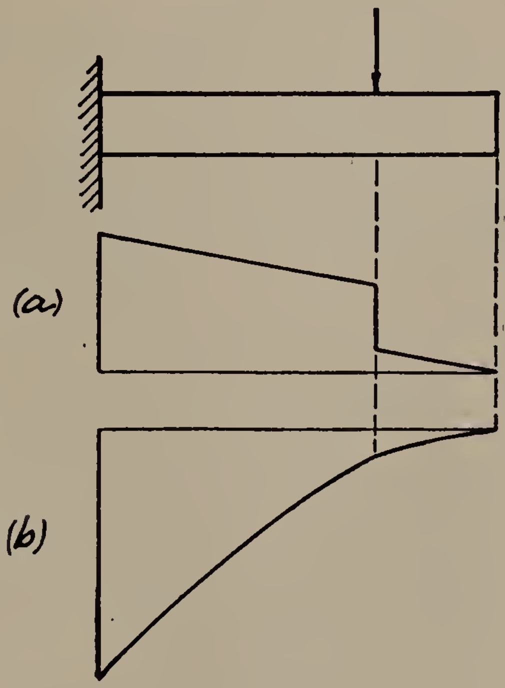


Fig. 74.

The shearing force and bending moment diagrams for a cantilever beam having a uniformly distributed load and a concentrated load is shown by Figs. 74*a* and 74*b*.

The maximum bending moment always comes at a point of zero shear except in the case of a cantilever beam when it comes at the fixed end.

Figure 75*a* shows the shearing force for a beam loaded at the center and supported at the ends. The same load is then spread over *a–a*, then *b–b*, *c–c*, *d–d*, *e–e* and then over the whole beam *g–g*.

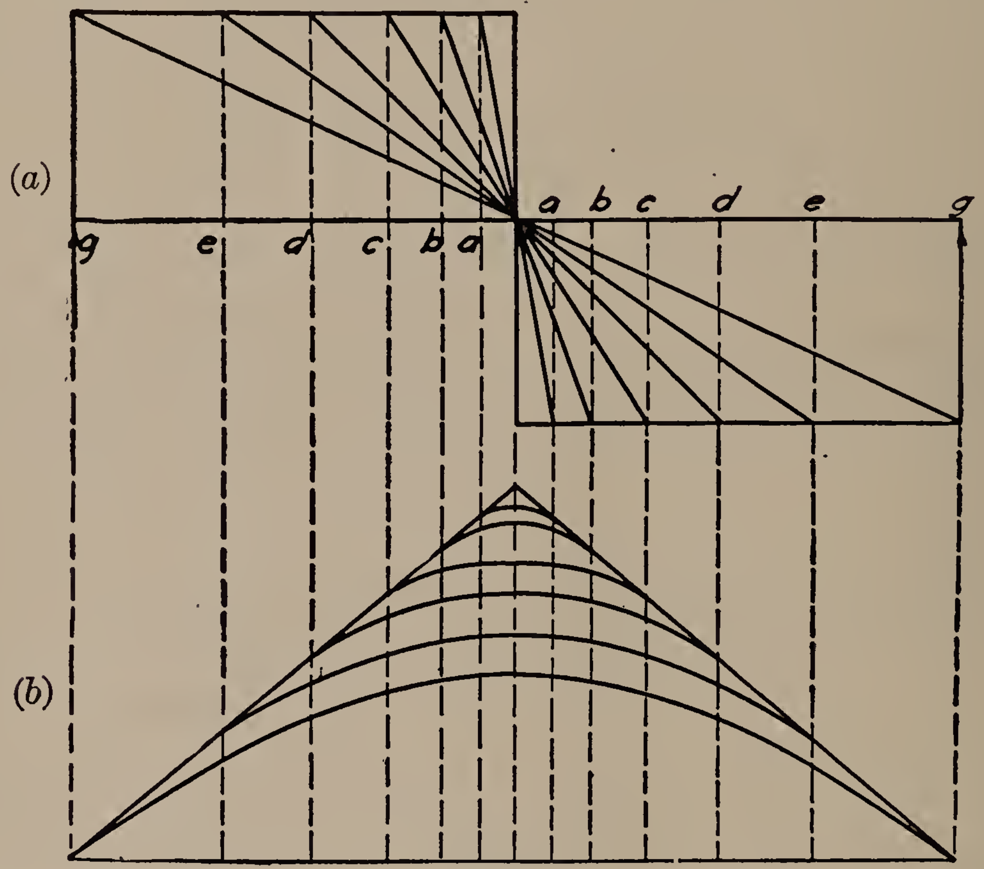


FIG. 75

Figure 75*b* shows the bending moments for the same loading; it will be noted that a distribution of load over a small distance makes very little difference in the magnitude of the bending moment.

Loads distributed over short distances may be treated as concentrated. Note that the bending moment for a concentrated load is twice that for the same load distributed over the whole beam, the beam being supported at the ends.

The shearing force and bending moment diagrams for an overhanging beam are shown in Fig. 76.

In this case there are two points of zero shear *b* and *c*. There are two possible points for maximum bending moment.

The point of inflection in a beam is the cross-section where the bending moment is zero and changes from positive to negative.

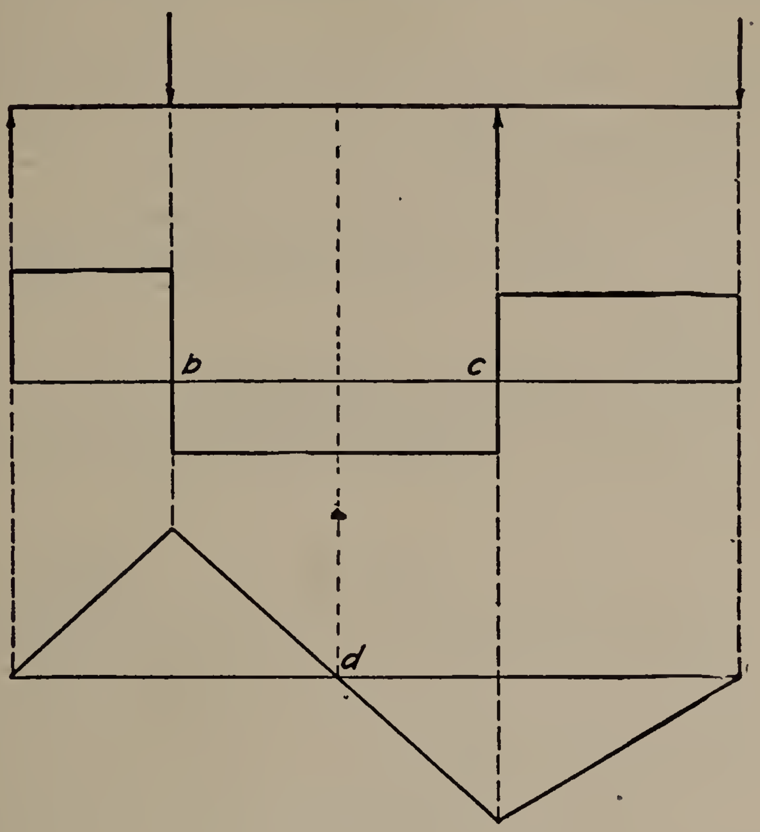


FIG. 76.

This point is also known as the point of contra-flexure and is shown at *d* (Fig. 76).

12. The maximum bending moments and shearing forces for four very common loadings are tabulated (Fig. 77).

a and *b* are fixed at one end.

c and *d* are freely supported at both ends.

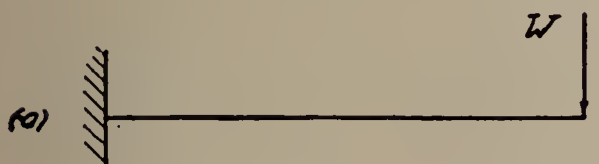
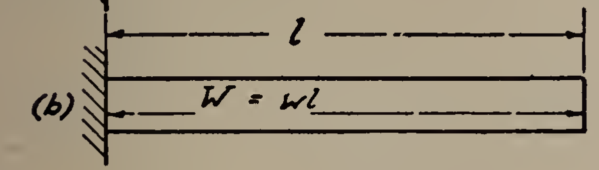
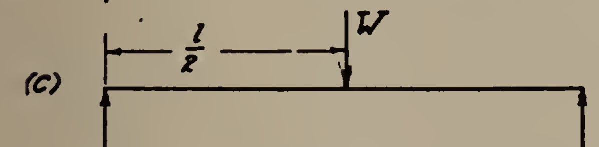
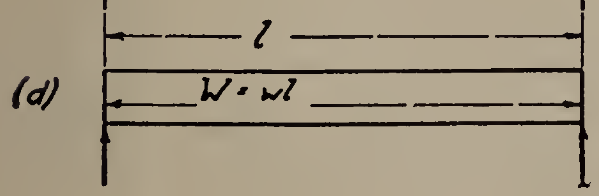
Loading	*Max. M*	*Max. S*
	Wl	W
	$\frac{Wl}{2} = \frac{wl^2}{2}$	$W = wl$
	$\frac{Wl}{4}$	$\frac{W}{2}$
	$\frac{Wl}{8} = \frac{wl^2}{8}$	$\frac{W}{2} = \frac{wl}{2}$

FIG. 77.

M is usually used to designate bending moment.
S is usually used to designate shearing force.

13. Problems. The formula $f = \frac{My}{I}$ is useful for the solving of four kinds of beam problems.

(1) Everything known but the fiber stress.
(2) Everything known but the dimensions of the cross-section of the beam.
(3) Everything known but the loading.
(4) Everything known but the length of the beam or location of load.

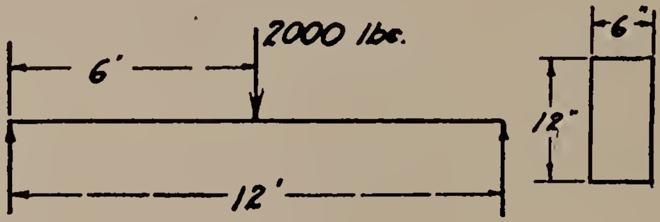


FIG. 78.

Problem Illustrating No. 1. Find maximum fiber stress (Fig. 78).

$$M_{\max} = \frac{Wl}{4} = \frac{2000 \times 12 \times 12}{4} \text{ inch pounds}$$

$$y = 6 \text{ inches}$$

$$I = \frac{6 \times 12 \times 12 \times 12}{12} \text{ inch units (usually designated ins.}^4\text{)}$$

Therefore

$$f = \frac{2000 \times 12 \times 12 \times 6 \times 12}{4 \times 6 \times 12 \times 12 \times 12} = 500 \text{ pounds per sq. inch.}$$

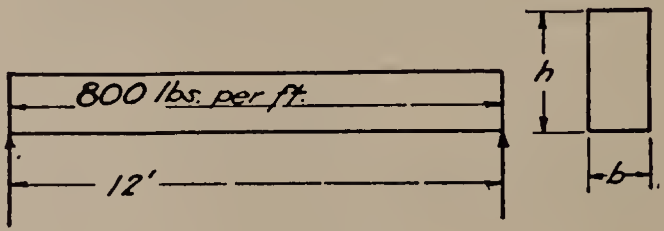


FIG. 79.

Problem Illustrating No. 2. Allowable fiber stress is 1200 pounds per square inch. Find proper rectangular cross-section (Fig. 79).

$f = \frac{My}{I}$. Therefore $\frac{M}{f} = \frac{I}{y}$, $\frac{I}{y}$ is known as the section modulus and is $= \frac{bh^2}{6}$ for a rectangular section.

$$M_{\text{max}} = \frac{Wl}{8} = \frac{800 \times 12 \times 12 \times 12}{8}$$

$$f = 1200$$

Therefore

$$\frac{M}{f} = \frac{800 \times 12 \times 12 \times 12}{8 \times 1200} = \frac{I}{y}$$

Therefore

$$\frac{I}{y} = 144 \text{ inch units usually designated In.}^3$$

A section must be chosen that will have $\frac{bh^2}{6}$ equal to or greater than 144 and must also be a commercial section.

Try 6″ × 10″, $\frac{I}{y} = \frac{6 \times 10 \times 10}{6} = 100$. This section is not large enough. Try a 6″ × 12″.

$$\frac{I}{y} = \frac{bh^2}{6} = \frac{6 \times 12 \times 12}{6} = 144$$

This section will be just large enough to carry the load but the weight of the beam also causes a stress, therefore a larger section must be used unless a slightly larger fiber stress is allowed.

The Handbooks give values for $\frac{I}{y}$ for all common rolled sections.

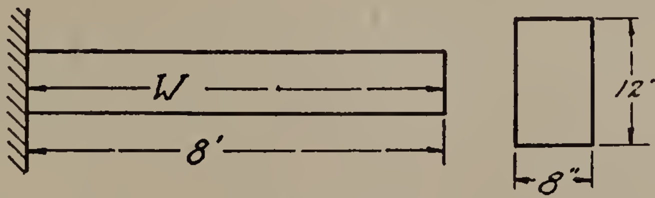


FIG. 80.

Problem Illustrating No. 3. If $f = 1200$ pounds per square inch, find the allowable load W uniformly distributed along the cantilever beam (Fig. 80).

$$f = \frac{My}{I} \qquad M = \frac{Wl}{2} \qquad \text{Therefore } W = \frac{2\,If}{ly}$$

$$\text{Therefore } W = \frac{2 \times 8 \times 12 \times 12 \times 12 \times 1200}{12 \times 8 \times 12 \times 6} = 4800 \text{ pounds}$$

Problem Illustrating No. 4. If $f = 1200$ pounds per square inch, find the length of beam (Fig. 81) that will safely support 1000 pounds per foot.

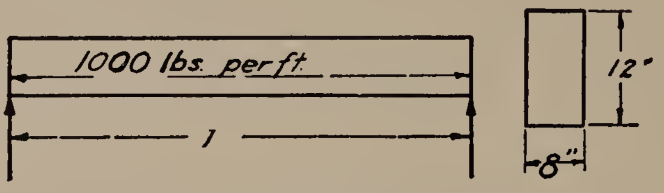


Fig. 81.

$$M = \frac{Wl}{8} = \frac{wl^2}{8} = \frac{1000 \times l^2}{8} \text{ where } l \text{ is measured in feet.}$$

$$f = \frac{My}{I} \quad \text{Therefore } 1200 = \frac{1000 \times l^2 \times 12 \times 6 \times 12}{8 \times 8 \times 12 \times 12 \times 12}$$

Therefore

$$l^2 = \frac{1200 \times 8 \times 8 \times 12 \times 12 \times 12}{1000 \times 12 \times 6 \times 12} = 153.6$$

Therefore

$$l = 12.4 \text{ feet}$$

It should be noted that the weight of the beam must always be taken into account when making calculations.

14. Floor Loads. Floor loads are usually considered to be uniformly distributed either over the whole floor or a part of it. The dead load is the weight of the floor itself with any permanent covering. The live load is the load that may be placed on the floor and may be due to merchandise, machinery, people, etc.

Problems: Steel I-beams are to be spaced 6 feet on centers and support a floor. Dead load is 50 pounds per square feet. Live load is 100 pounds per square foot. Allowable fiber stress is 16,000 pounds per square inch. Find suitable beam. The span of the beams is 16 feet.

The total load on any one beam is $150 \times 16 \times 6 = 14{,}400$ pounds. It is usually assumed that a beam supports the floor half way to the next beam. This load is uniformly distributed. Therefore

$$\text{Maximum bending moment is } \frac{Wl}{8} = \frac{14{,}000 \times 16 \times 12}{8} \text{ inch pounds}$$

Therefore

$$\frac{I}{y} = \frac{M}{f} = \frac{14{,}400 \times 16 \times 12}{8 \times 16{,}000} = 21.6 \text{ In.}^3$$

The Carnegie Handbook under elements of structural beams gives the section modulus of a 9″–30 pound beam as 22.5 In.^{3} This will be a suitable section.

Find the live load that can be placed upon a floor supported by wooden beams 6″ × 12″ section and 15 foot span, spaced 5 feet on centers. Dead load is 50 pounds per square foot and allowable fiber stress is 1200 pounds per square inch.

These beams are loaded with uniformly distributed load. Therefore

$$W = \frac{8\,If}{ly} \text{ where } W \text{ is the total load}$$

Therefore

$$W = \frac{8 \times 6 \times 12 \times 12 \times 12 \times 1200}{12 \times 15 \times 12 \times 6} = 7680 \text{ pounds}$$

7680 divided by the square feet of floor supported by one beam equals $\frac{7680}{5 \times 15} = 102.4$ pounds per square foot total load.

$$102.4 - 50 = 52.4 \text{ pounds per square foot}$$

Therefore 52.4 pounds per square foot is safe live load.

15. Deflection. The formulae for deflection that are useful when the limitations of the common beam theory hold true are the result of the application of the calculus to the theory of beams. The radius of curvature for any curve is

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Let x be the distance measured along the axis of a beam from any origin and y be the deflection up or down. Substitute v for y in the formula. From the first assumption of the theory of beams, see page 65, *e*, the strain on any fiber of a beam was found to be $\frac{1}{r} \times y$ where r is the radius of curvature of the neutral line of the beam.

Therefore $$\frac{1}{r}=\frac{e}{y}=\frac{\dfrac{d^2v}{dx^2}}{\left[1+\left(\dfrac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

For any beam the deflection is small and, therefore, the slope $\frac{dv}{dx}$ must also be very small and the denominator of the right hand member of the equation will be approximately unity.

Therefore $$\frac{d^2v}{dx^2}=\frac{e}{y}\qquad \text{But } e=\frac{p}{E}$$

Therefore $$\frac{d^2v}{dx^2}=\frac{p}{Ey}\qquad \text{But } \frac{p}{E}=\frac{M}{I}$$

Therefore $$\frac{d^2v}{dx^2}=\frac{M}{EI}$$

M can be expressed in terms of x; E and I are constants.

If $$\frac{d^2v}{dx^2}=\frac{M}{EI}$$

The slope $$\frac{dv}{dx}=\int\frac{M}{EI}dx$$

And the deflection $$v=\int\int\frac{M}{EI}dx^2$$

The constants of integration must be determined.

16. The application of this formula to the simple case of a beam supported at the ends and loaded with a uniformly distributed load will be given.

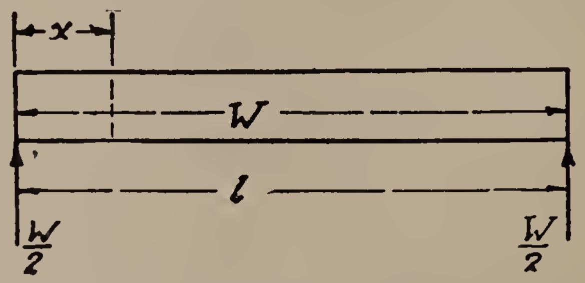


Fig. 82.

Assume an origin at the left end of the beam. The bending moment, distance x from the point, is

$$\frac{Wx}{2}-\frac{W}{l}\frac{x^2}{2}$$

Therefore $$\frac{d^2v}{dx^2} = \left[\frac{W}{2}x - \frac{W}{l}\frac{x^2}{2}\right]\frac{1}{EI}$$

$\frac{dv}{dx}$ is the slope of the curve of the neutral line of the beam. Therefore the slope is

$$\frac{1}{EI}\int\left[\frac{Wx}{2} - \frac{W}{l}\frac{x^2}{2}\right]dx = \frac{1}{EI}\left[\frac{Wx^2}{4} - \frac{Wx^3}{6l}\right] + c_1$$

Note that when $x = \frac{l}{2}$, the slope is zero. Evaluate for c_1 by using this condition.

Slope $= 0 = \frac{1}{EI}\left[\frac{Wl^2}{16} - \frac{Wl^3}{48l}\right] + c_1$. Therefore $c_1 = \frac{Wl^2}{24EI}$.

Therefore the general formula for slope anywhere in the beam is

$$\frac{1}{EI}\left[\frac{Wx^2}{4} - \frac{Wx^3}{6l} - \frac{Wl^2}{24}\right]$$

Integrate again and obtain deflection (v)

$$v = \frac{1}{EI}\left[\frac{Wx^3}{12} - \frac{Wx^4}{24l} - \frac{Wl^2x}{24}\right] + c_2$$

Note that when $x = 0$, $v = 0$.
Use this condition to obtain $c_2 = 0$. Therefore the general expression for deflection is:

$$\frac{1}{EI}\left[\frac{Wx^3}{12} - \frac{Wx^4}{24l} - \frac{Wl^2x}{24}\right]$$

The deflection is a maximum when $x = \frac{l}{2}$. Substitute this value in the general expression and obtain $-\frac{5}{384}\frac{Wl^3}{EI}$. The minus sign means that the deflection is down. The derivations of other formulae for deflection are carried out in a similar manner and for complicated loads become long and laborious. The formulae required for the solution of the majority of the problems met in practice can be found listed in any handbook.

It should be noted that the differentiation of a general expression for deflection will be a general expression for slope. Therefore, placing a general expression for slope equal to zero may be valuable in the determination of maximum deflection.

The maximum deflection for a beam fixed at one end and loaded at the other is $\frac{1}{3}\frac{Wl^3}{EI}$, when loaded uniformly the maximum deflection is $\frac{1}{8}\frac{Wl^3}{EI}$ where W is the total load. For beams supported at the ends, the maximum deflection when loaded at the center is $\frac{1}{48}\frac{Wl^3}{EI}$ and when loaded uniformly with load W, $\frac{5}{384}\frac{Wl^3}{EI}$.

17. Problems. Determine the maximum deflection for the beam loaded as shown in Fig. 83.

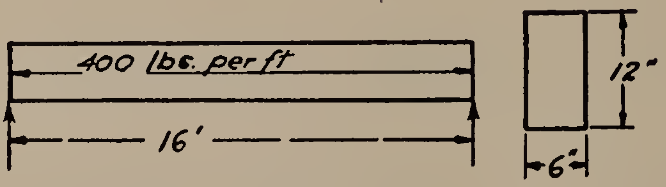


FIG. 83.

$E = 1{,}200{,}000$ pounds per square inch. Maximum deflection for this case is $\frac{5}{384}\frac{Wl^3}{EI}$ where W is total load.

The dimension must be all in inches or all in feet. Using inches

$$V_{\max} = \frac{5 \times 400 \times 16 \times 16 \times 16 \times 16 \times 12 \times 12 \times 12 \times 12}{384 \times 1{,}200{,}000 \times 6 \times 12 \times 12 \times 12}$$
$$= 0.569 \text{ inches}$$

Using feet

$$V_{\max} = \frac{5 \times 400 \times 16 \times 16 \times 16 \times 16 \times 12 \times 144 \times 144}{384 \times 1{,}200{,}000 \times 144 \times 6 \times 12 \times 12 \times 12}$$
$$= 0.0474 \text{ feet}$$
$$= .0474 \times 12 = 0.569 \text{ inches as before.}$$

The moment of inertia of an area in foot units is equal to the moment of inertia in inch units divided by 12 to the fourth power.

The modulus of elasticity in pounds per square foot is equal to the modulus of elasticity in pounds per square inch multiplied by 144.

The maximum allowable deflection in most beams is $\frac{1}{360}$ of length and in some cases it is kept down to $\frac{1}{400}$ of length.

Find the concentrated load that can be placed at the center of a beam 6″ × 12″ cross-section and 18 feet long without causing a deflection of more than $\frac{l}{360}$, $E = 1{,}200{,}000$.

$$v = \frac{l}{360} = \frac{18 \times 12}{360} = \frac{1 \times W \times 18 \times 18 \times 18 \times 12 \times 12 \times 12 \times 12}{48 \times 1{,}200{,}000 \times 6 \times 12 \times 12 \times 12}$$

Therefore $W = 2963$ pounds.

In the design of beams due consideration must be given to both maximum allowable fiber stress and to deflection. Take for example a beam supported at the ends and loaded uniformly. For any cross-section, there is a certain length below which fiber stress will determine the design and above which deflection will be the determining factor.

Consider a wooden beam 6″ × 12″ section. Maximum allowable fiber stress 1200 pounds per square inch. Maximum allowable deflection $\frac{l}{360}$. $E = 1{,}200{,}000$ pounds per square inch. The load is uniformly distributed. The formula for fiber stress is $f = \frac{My}{I}$, that for deflection is $\frac{5}{384}\frac{Wl^3}{EI}$, $M = \frac{Wl}{8}$. Therefore $W = \frac{8\,If}{ly}$. Substitute this value in the formula for deflection and solve for l. This value for l will be a length where if the load is such that the maximum fiber stress is 1200 pounds per square inch, the maximum deflection will be $\frac{l}{360}$.

$$\frac{l}{360} = \frac{5 \times 8\,Ifl^3}{384\,lyEI} = \frac{5 \times 8 \times 1200\,l^3}{384\,l \times 6 \times 1{,}200{,}000}.$$

Therefore $l = 160$ inches $= 13\frac{1}{3}$ feet.

If l is more than $13\frac{1}{3}$ feet, the beam cannot be stressed to 1200 pounds per square inch without causing a deflection of more than $\frac{l}{360}$. If the beam is less than $13\frac{1}{3}$ feet long, the deflection cannot be $\frac{l}{360}$ without stressing the beam to more than 1200 pounds per square inch.

The formulae for the deflection of beams subjected to complicated systems of loading are complex and the exact solution of such problems is a long process. Many approximate methods are used which give safe beams although the maximum deflection is not actually found.

18. Longitudinal Shear. Beams of ordinary dimensions that are all one piece are usually designed by either fiber stress or deflection. But when the loads are applied in such a manner as to cause large shearing stresses with comparatively small bending moments another stress must be investigated. If a beam is divided into two parts by a longitudinal plane not parallel to the plane of loading there is usually a tendency for one part of the beam to slide by the other longitudinally. This tendency is resisted by the strength of the material in shear.

Beams that are built up of several parts bolted or riveted together come under this class. The different parts would slide by each other if this action was not prevented by the rivets or bolts.

The beam in Fig. 84 is supported at the ends and loaded as shown. Consider two sections *a–a* and *c–c*. Let the bending moment at *a–a* be M_a and the bending moment at *c–c* be M_c. Take the small block *acdo* in the beam, *do* being parallel to the axis of the beam. This block will have two sides *acdo*, a width of *b* and a thickness of *t*. The faces *ad* and *co* will be acted upon by a uniformly varying stress the intensity of which at any point is $f = \frac{My}{I}$. If $M_a = M_c$, the resultant of each of these uniformly varying stresses will be of the same magnitude and there will be no tendency to move the block, but, if M_a does not equal M_c, the resultants will not be of equal magnitude and there will be a tendency to move the block in one direction or the other. This force will be resisted by the shearing strength of the material along

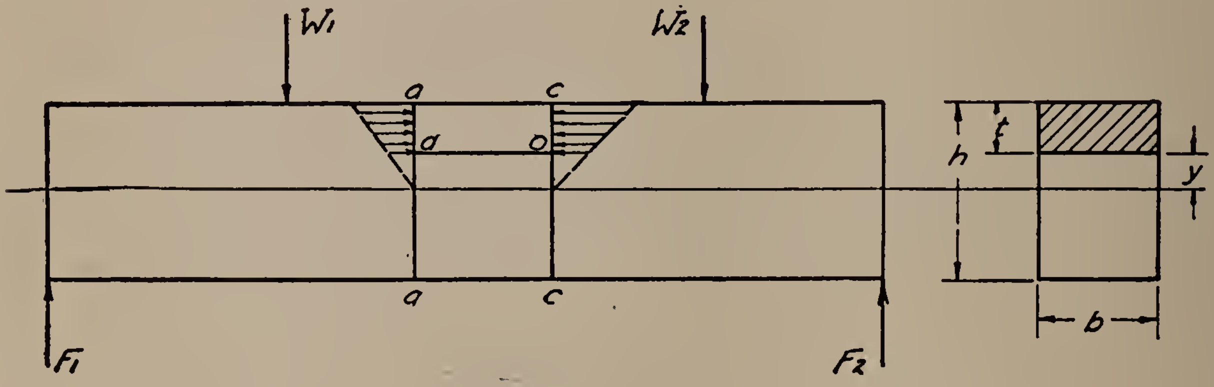


Fig. 84.

the bottom surface of the block where it joins the rest of the beam. This surface is *do* long and *b* wide.

Let the resultant force on surface *ad* equal R_a.

Let the resultant force on surface *co* equal R_c.

The force that will tend to move the block is $R_a - R_c$ or $R_c - R_a$, whichever of the two is the larger. Call this force ΣR. Divide ΣR by the area *do* times *b* and the average shearing force at the base of the block in the material of the beam will be found. For any definite case the average shear can be determined. A general formula for the intensity of this shearing stress can be derived as follows: (Fig. 85).

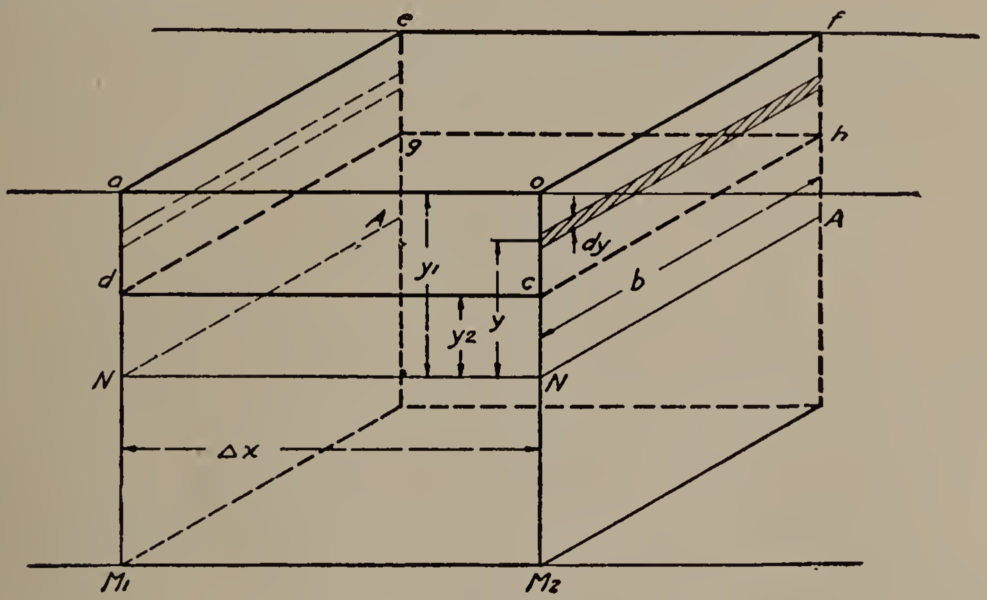


FIG. 85.

Let M_1 and M_2 be the bending moments at two sections of a beam Δx apart. *NA* is the neutral axis of either cross-section. Consider a block in the beam. The upper surface of this block is *aofe* and is at the surface of the beam, the bottom of the block is *dchg* and is parallel to the neutral layer of the beam. The sides lie in the cross-sections where the bending moments are M_1 and M_2. The ends of the block are at the outside of the beam. The forces acting on this block must be in equilibrium. The only forces acting that will tend to move the block horizontally are the normal stresses on the faces *ochf* and *adge*. They are balanced by the resisting shearing force on the face *dchg*.

The normal force on face *ochf* is $\frac{M_2}{I}\int_{y_2}^{y_1} yb\,dy$.

M_2 is the bending moment at the section. The intensity of normal stress anywhere at the section is $\frac{M_2 y}{I}$. The stress upon a strip *dy* wide across the section parallel to *NA* is $\frac{M_2 y}{I} b\,dy$ and for the whole surface *ochf* the integral as indicated. In the same manner the stress on the face *adge* may be found to be

$$\frac{M_1}{I}\int_{y_2}^{y_1} yb\,dy$$

The difference between these stresses is

$$\frac{\Delta M}{I}\int_{y_2}^{y_1} yb\,dy$$

Δ (delta) means difference. Divide this difference of stresses by the area of the base of the block and obtain an expression for average shear intensity.

$$\frac{\dfrac{\Delta M}{I}\displaystyle\int_{y_2}^{y_1} yb\,dy}{b\,\Delta x} = \text{average intensity of longitudinal shear.}$$

To obtain the exact intensity of shear, the area *dchg* must be made to approach zero as a limit. This can be accomplished by making Δx become dx in which case ΔM will become dM and the expression for intensity of shear will be:

$$\frac{\dfrac{dM}{dx}\displaystyle\int_{y_1}^{y_2} yb\,dy}{bI}$$

$\frac{dM}{dx}$ is equal to the shearing force at the section, $\int_{y_1}^{y_2} yb\,dy$ is the moment of the area corresponding to *ochf* about *NA* and *I* is the moment of inertia of the area of [the whole cross-section about *NA*. The formula then reduces to $\frac{SQ}{bI}$.

where S = Shearing force at the section.

The fact that $\frac{dM}{dx}$ = shearing force can be shown as follows:

AA (Fig. 86) is any section of a beam. The loads on the left of the section are as shown. Section *AA* is distance x from the point *O* which is taken as the origin.

The bending moment at AA is:

$$-W_1(x-a_1)-W_2(x-a_2)-\frac{wx^2}{2}+Fx=M$$

$$\frac{dM}{dx}=-W_1-W_2-wx+F=\text{shearing force at } AA=S$$

It should be noted that in the derivation of $\frac{SQ}{bI}$ the cross-section of the beam was divided into two parts by a line parallel to the neutral axis. If the beam had been of varying section, the same formula could have been obtained by expressing the width of the elementary strip as a function of y.

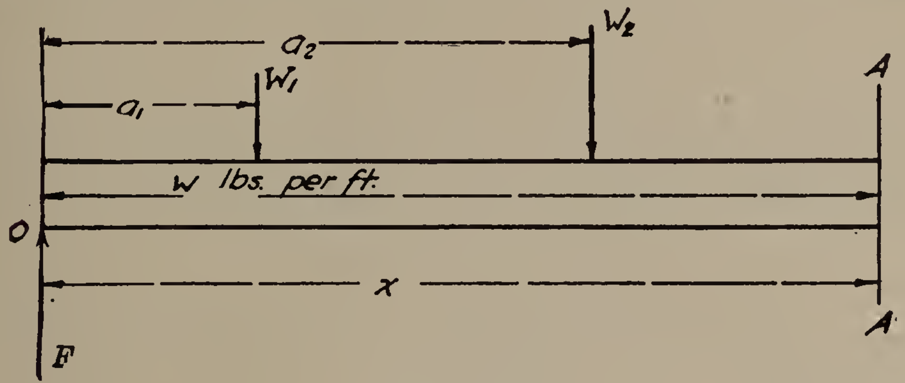


FIG. 86.

S = shearing force at section.

Q = moment of one of the parts into which the cross-section is divided, about the neutral axis.

b = width of the beam at the place where the intensity of longitudinal shearing stress is desired.

I = moment of inertia of the cross-section about the neutral axis. The neutral axis passes through the center of gravity of the section and at right angles to the plane of loading.

General Method for Determining the Maximum Intensity of Longitudinal Shear.

Find section where the shearing force is a maximum.

Determine the point in the section where $\frac{Q}{b}$ is the largest.

$\frac{Q}{b}$ is usually the largest at the neutral axis but not always. In ordinary beams of rectangular or I section, the maximum intensity of longitudinal shear is at the neutral axis, but in beams where the material is concentrated near the neutral axis, the maximum intensity may be at some distance out.

19. Problems. Find the maximum intensity of longitudinal shear in the beam shown in Fig. 87.

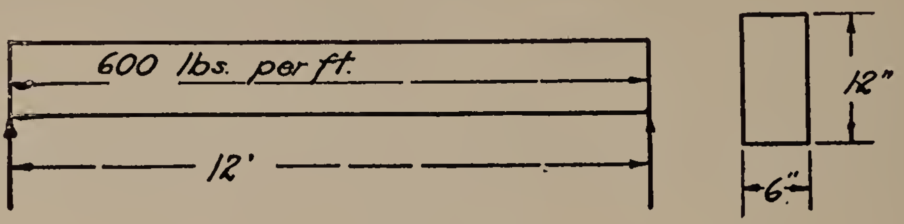


FIG. 87.

The maximum shearing force is 3600 pounds (at the ends). The maximum Q is at the neutral axis and is

$$6 \times 6 \times 3 = 108$$

$$I \text{ is } \frac{6 \times 12 \times 12 \times 12}{12} = 864 \text{ In.}^4$$

b is constant and equal to 6 inches.

Let s = intensity of longitudinal shear.

$$\text{Maximum } s = \frac{SQ}{bI} = \frac{3600 \times 108}{6 \times 864} = 75 \text{ pounds per square inch.}$$

Longitudinal shear is an important factor in beams loaded as shown (Fig. 88).

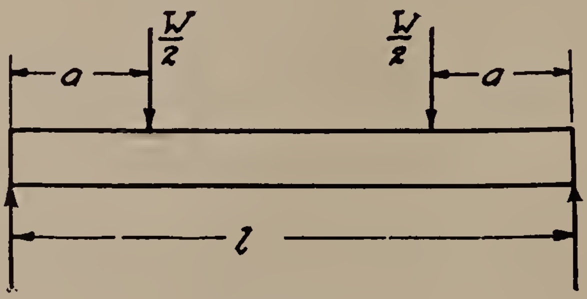


FIG. 88.

The maximum bending moment is $\frac{Wa}{2}$ and the maximum shear is $\frac{W}{2}$, if a is small the allowable intensity of longitudinal shear will design the beam.

Assume the cross-section of the beam, Fig. 88, to be 6″ × 12″ and a = 1 foot, l = 12 feet.

Allowable fiber stress is 1200 pounds per square inch.

Allowable longitudinal shear is 100 pounds per square inch.

Determine the load that will be safe when the fiber stress alone is considered,

$$1200 = \frac{W \times 1 \times 12 \times 6 \times 12}{2 \times 6 \times 12 \times 12 \times 12} \quad \text{Therefore } W = 28{,}800 \text{ pounds}$$

Now determine the load that will be safe when longitudinal shear is considered.

$$100 = \frac{W \times 6 \times 6 \times 3 \times 12}{2 \times 6 \times 6 \times 12 \times 12 \times 12}. \quad \text{Therefore } W = 9600 \text{ pounds.}$$

It will be noted that longitudinal shear determines the load that this beam can safely carry.

20. When a beam is built up and bolted as shown (Fig. 89) a modification of the formula for longitudinal shear must be made.

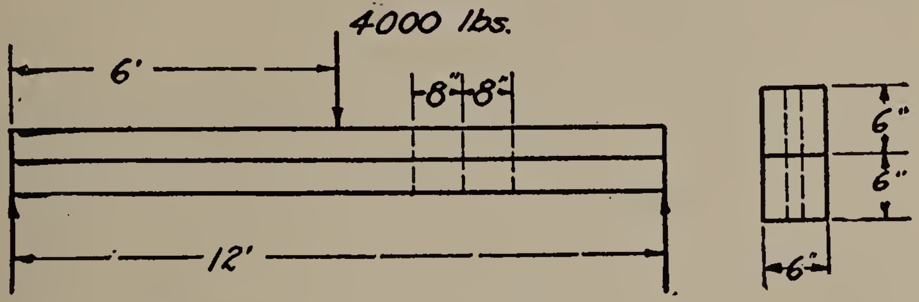


FIG. 89

The beam, Fig. 89, is made of two 6″ × 6″ pieces bolted together. Assume that the bolts are 8 inches apart, determine the shear at the middle section of a bolt. $\frac{SQ}{bI}$ is the intensity of longitudinal shear per square inch. If this value is multiplied by b, the longitudinal shear per inch of length of the beam is obtained and it makes no difference how the cross-section is divided.

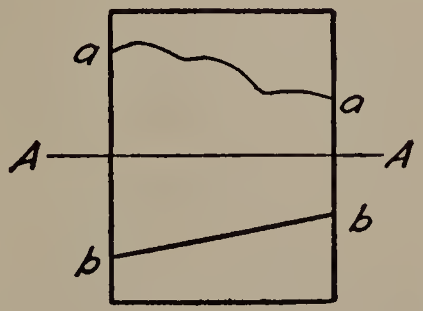


FIG. 90.

The longitudinal shear per one inch in length of the beam the cross-section of which is shown in Fig. 90, can be found for the line aa or bb. If the intensity per square inch was wanted, the cross-section must be divided by a line parallel to the neutral axis. Returning to the beam of Fig. 89, the longitudinal shear per unit of length of the beam can be found by applying the formula

$$\frac{SQ}{I} = \frac{2000 \times 6 \times 6 \times 3 \times 12}{6 \times 12 \times 12 \times 12} = 250 \text{ pounds per unit of length}$$

It will be noted that the stress on all the bolts will be the same because the shearing force at all sections except at the center is the same. Assume that each bolt provides for the shear half way to the next bolt on either side. One bolt then provides the resistance necessary for 8 units of length and, therefore, the shear at the middle section of each bolt is 8 × 250 = 2000 pounds.

Note that the bolt simply does what solid wood would do if the beam was all one piece. If the load was uniformly distributed, the shearing force would be the greatest at the ends and diminish uniformly to zero at the center. If the bolts were uniformly spaced in such a case the shearing stress would decrease in the bolts towards the center, being a maximum in the bolts near the ends. Sometimes advantage is taken of this and the pitch of the bolts is increased towards the center for uniformly distributed loads.

Steel beams are often built up from rolled sections in which case the longitudinal shear becomes an important factor in the spacing of the rivets.

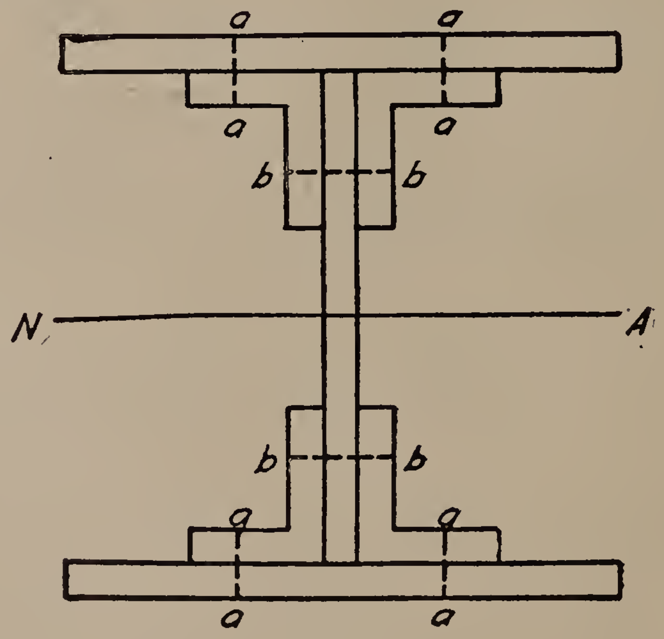


FIG. 91.

Consider the cross-section (Fig. 91). This beam is built up with a web plate, two flange plates and four angles. The flange rivets, *aa*, are driven in pairs and the web rivets, *bb*, are driven in a single row.

The allowable bearing pressure for the rivets on the plate is known also the allowable shearing strength of the rivets. Determine the bearing and shearing value for each rivet. The rivets, *aa*, are in single shear, the rivets, *bb*, are in double shear. The smaller of the two resistances is used for the design. Assume that the diameter of the rivets is known. Determine the longitudinal shearing resistance that must be provided per unit of length by rivets, *aa*, using $\frac{SQ}{I}$. S is the vertical shear at the section under discussion (usually where S is a maximum), Q is the moment of the area of the cross-section of the flange plate about NA. I is the moment of inertia of the whole section about NA.

Divide the least resistance offered by the two rivets by the value of $\frac{SQ}{I}$. The result will be the pitch required.

For the web rivets *bb*. These rivets prevent the flange (plate and angles) from sliding upon the web; therefore, Q is the moment of the area of the cross-section of the flange plate and angles about NA. S and I are the same as before. The pitch is found as in the case of the flange rivets. The actual spacing of rivets in a girder depends upon many other factors besides longitudinal shear.

21. Relation between Shearing Stresses. One very important fact with regard to shearing stresses is that the intensity of the shearing stress on any two planes at right angles to each other is the same.

abcd (Fig. 92) is a particle of unit thickness in a beam; the forces acting upon the particle must be in equilibrium. The normal forces are balanced. Now consider the shearing forces. Assume that the forces all act parallel to the paper.

Let f_1 be the intensity of the shearing stresses upon faces *ab* and *dc*.

Let f_2 be the intensity of the shearing stresses upon faces *ad* and *bc*.

The shearing force acting on *ab* is $ab\,f_1 = cdf_1$.

The shearing force acting on *ad* is $ad\,f_2 = bcf_2$.

abf_1 and cdf_1 form a right-handed couple whose moment is $abf_1 \times ad$.

adf_2 and bcf_2 must form a left-handed couple whose moment is $adf_2 \times ab$. ΣM must be zero.

Therefore $abf_1 \times ad = adf_2 \times ab.$

Therefore $f_1 = f_2$

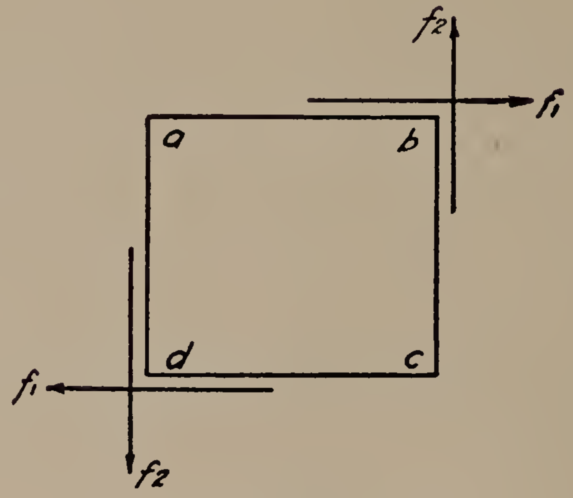


Fig. 92.

From this demonstration it will be noted that the intensity of vertical shear at any point in the cross-section of a beam is equal to the intensity of the longitudinal shear at the same point. Both of these are zero at the outside of the beam.

An interesting example of the use of this fact is found in the web of a plate girder (Fig. 93).

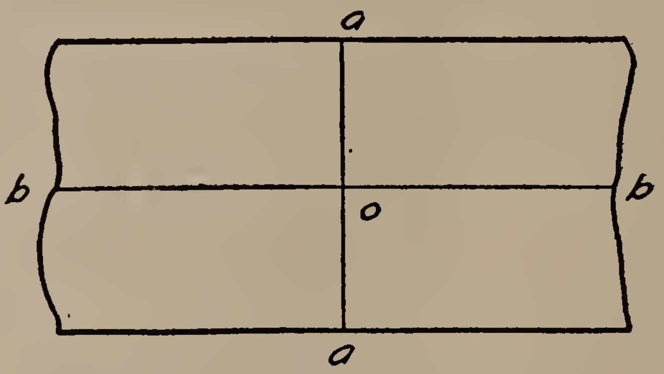


Fig. 93.

aa is drawn at right angles to *bb* which is the axis of the beam. There is no tension or compression at *O* on plane *aa, O* being in the neutral layer of the beam. The longitudinal and vertical shear are the only stresses acting at point *O* on the planes *aa* and *bb*. These stresses are of the same intensity. Consider a small particle at the center of the beam of a shape shown (Fig. 94).

f is the intensity of shear on *ab* and *oa*. The force acting on *oa* is $oa \times f$ and that acting on *ob* is $ob \times f$. The particle must be in equilibrium, therefore the force acting upon *ab* must balance the resultant of $f \times oa$ and $f \times ob$. If the angle oba = angle bao = 45°, f_1 will be normal to ab and act as indicated by the arrow. This shows that at the neutral axis of a plate girder, there may be considerable compression on a plane making 45° with the axis of the beam. It needs very little study to realize that this compression is liable to become a factor in the design of the beam as the compression will tend to buckle the web. The placing of web stiffeners depends largely upon this fact. It can be shown that tension will exist upon a plane at right angles to ab (Fig. 94). This fact is of great importance in the design of reinforced concrete beams where steel must be placed wherever tension occurs.

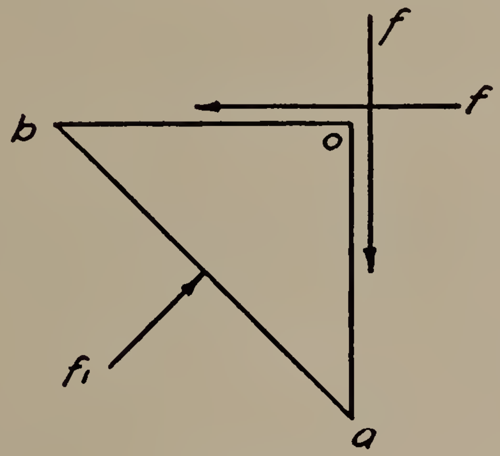


FIG. 94.

In beams of rectangular cross-section the distribution of shear is such as to make it impossible to use the average vertical shear in design. In an I section the vertical shear may be assumed to be uniformly distributed over the web with comparatively little approximation.

22. Design of Beams. We now have all the formulae necessary for the design of a common beam.

$$f = \frac{M}{I} \text{ for fiber stress}$$

$$s = \frac{SQ}{bI} \text{ for longitudinal shear}$$

and the various formulae for maximum deflection.

A beam must be so proportioned that the allowable unit stresses for tension, compression and shear will not be exceeded and the maximum deflection not more than a specified amount.

Sometimes one of the three is the determining factor and sometimes another.

Problem: Determine the safe load for the beam (Fig. 95).

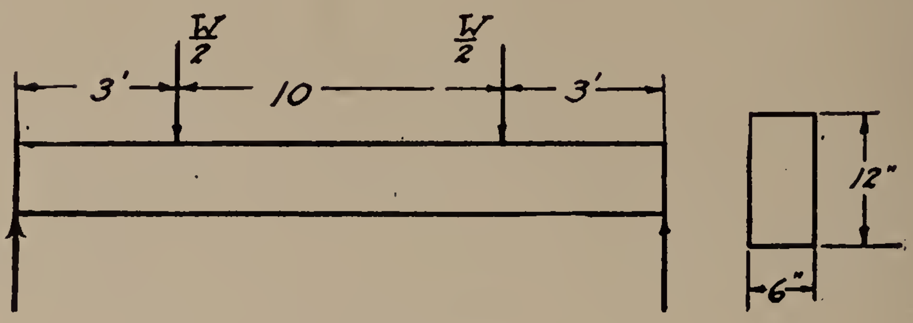


FIG. 95

Given:

$$f = 1200 \text{ pounds per square inch}$$
$$s = 100 \text{ pounds per square inch}$$
$$\text{Max. } v = \frac{l}{360}$$
$$E = 1{,}200{,}000 \text{ pounds per square inch}$$

Find safe W.

Fiber Stress $$f = \frac{My}{I}$$

$$1200 = \frac{W \times 3 \times 12 \times 6 \times 12}{2 \times 6 \times 12 \times 12 \times 12}$$

Therefore $$W = 9600 \text{ pounds}$$

Longitudinal Shear $$S = \frac{SQ}{bI}$$

$$100 = \frac{W \times 6 \times 6 \times 3 \times 12}{2 \times 6 \times 6 \times 12 \times 12 \times 12}$$

Therefore $$W = 9600 \text{ pounds}$$

The Maximum Deflection for this case is,

$$\frac{Wa}{12\,EI}\left(\tfrac{3}{4}\,l^2 - a^2\right) \text{ where } a = 3 \text{ feet and } l = 16 \text{ feet}$$

$$\frac{16 \times 12}{360} = \frac{W \times 3 \times 12}{12\,EI}\left[\frac{3}{4} \times 16^2 \times 144 - 3^2 \times 144\right]$$

Therefore

$$\frac{16 \times 12}{360} = \frac{W}{48 \times 1{,}200{,}000 \times 6}\left[26{,}352\right]$$

Therefore $$W = 6995 \text{ pounds}$$

The load that this beam will carry is determined by the deflection.

If the length is reduced, the deflection will become less and less of a factor.

23. Constrained Beams. Beams that are more or less constrained at both ends are often met in construction.

Beams of this kind will not be taken up here as the cases are very uncertain and their solution depends very largely upon practice. Formulae for use with such beams can be readily found in handbooks.

24. Limitations of Common Theory Not Fulfilled. Up to this point it has been assumed that all the limitations of the common beam theory have been fulfilled. There are many cases where these limitations are departed from and certain of them will be taken up briefly.

In what follows all of the limitations but one will be assumed to hold in any particular case.

Forces not Perpendicular to Axis of Beam. (Fig. 96.)

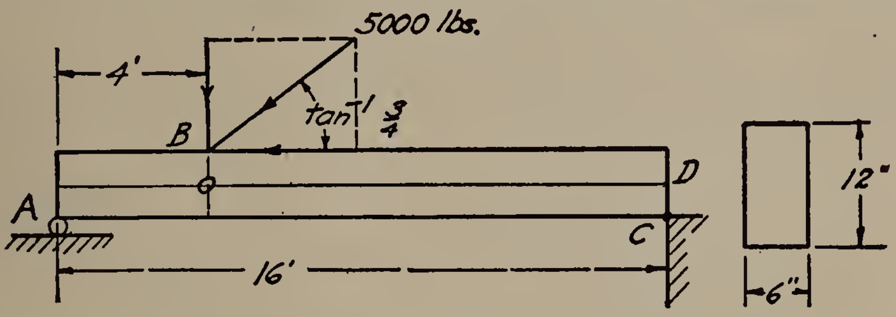


Fig. 96.

Problem: The beam is supported by a roller at A and a pin at C. The beam is loaded with 5000 pounds acting at B as shown.

Resolve the load into horizontal and vertical components at the point B and determine the supporting forces. The support at A is vertical on account of the roller. The horizontal component of the support at C is 4000 pounds and acts on the beam to the right. The vertical component of the support at C is 500 pounds. The support at A 2500 pounds.

The horizontal force of 4000 pounds at B will have the same effect as a force equal and parallel to it at O and a left-handed couple whose moment is $4000 \times \frac{1}{2} = 2000$ foot-pounds.

The horizontal force of 4000 pounds acting at C will have the same effect on a force equal and parallel to it at D and a left-handed couple whose moment is $4000 \times \frac{1}{2} = 2000$ foot-pounds.

Separate the forces and couples into two groups, one of which will cause bending stresses and the other direct stresses. The group causing bending stresses is shown by Fig. 97.

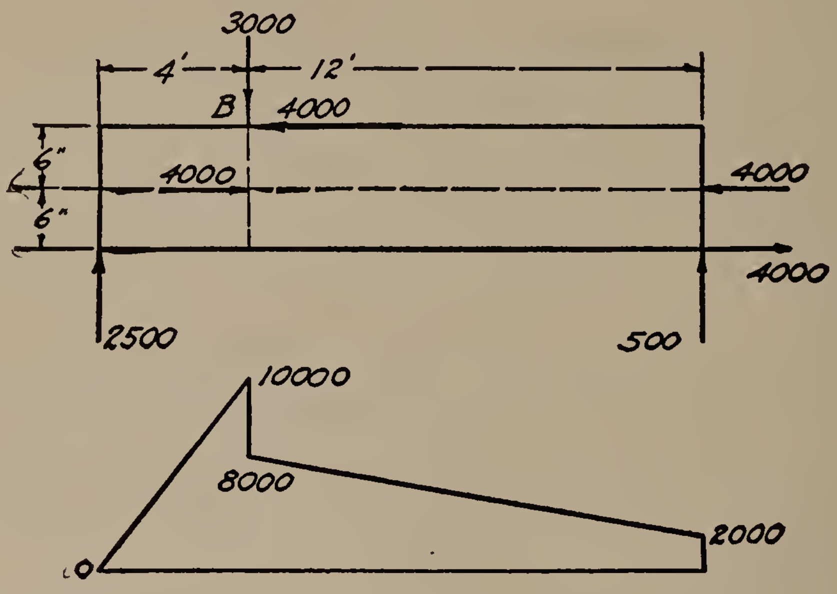


FIG. 97.

Note that the bending moment at the right end is not zero and the sudden change in the bending moment at B due to the action of the couple.

The group of forces causing direct stresses is shown by Fig. 98.

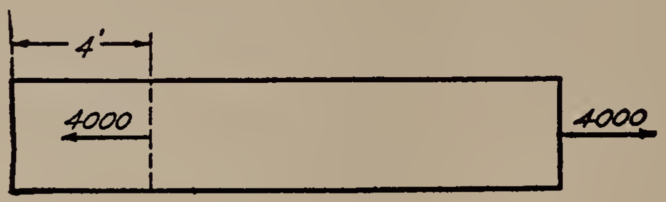


FIG. 98.

There is a direct stress of tension between B and D equal to $\frac{4000}{72} = 55$ pounds per square inch.

From Fig. 97 the bending stresses can be determined. Just to the left of B the bending moment is 10,000 foot-pounds, giving

$$f = \frac{My}{I} = \frac{10{,}000 \times 12 \times 6 \times 12}{6 \times 12 \times 12 \times 12} = 833 \text{ pounds per square inch.}$$

Just to the right of B the bending moment is 8000 foot-pounds, giving

$$f = \frac{My}{I} = \frac{8000 \times 12 \times 6 \times 12}{6 \times 12 \times 12 \times 12} = 667 \text{ pounds per square inch}$$

In addition to this, there is a direct stress of 55 pounds per square inch tension over the area making the maximum intensity of the combined stress 667 + 55 = 722 pounds per square inch.

The maximum stress intensity in this case is just to the left of section *B*, and is tension.

Sometimes the direct stress is large enough to more than offset the difference in bending stresses.

Columns loaded eccentrically come under this case.

The column shown in Fig. 99 is loaded vertically with 14,400 pounds. If the load was at the center of the column, the intensity of the stress over the cross-section would be uniform and equal to

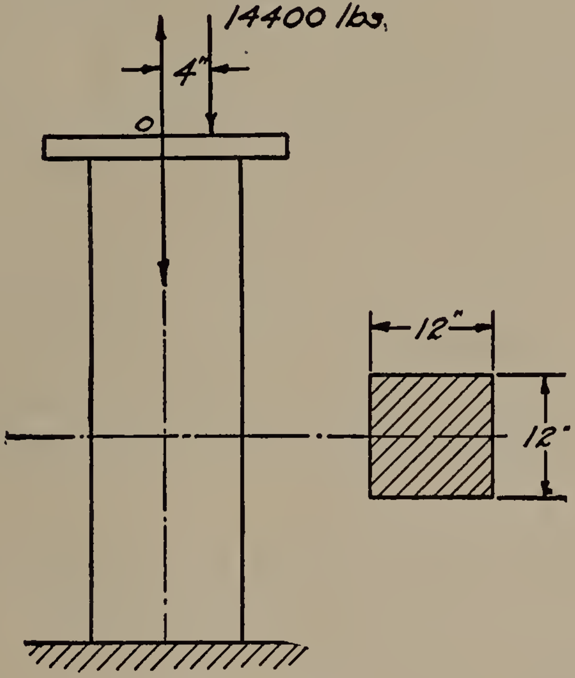


Fig. 99.

$$\frac{14{,}400}{12 \times 12} = 100 \text{ pounds per square inch}$$

If the load is applied 4 inches from the center of the column, there will be a combined stress due to the direct and bending stresses.' Resolve the 14,400 pound force into a force equal and parallel to it acting at the center of the column and a right-handed couple, the moment of which is 14,400 × 4 = 57,600 foot-pounds. The downward force at *a* will cause a uniformly distributed compression over the cross-section of 100 pounds per square inch. The couple will cause a bending in the column. The maximum fiber stress due to this bending will be $f = \frac{57{,}600 \times 6 \times 12}{12 \times 12 \times 12 \times 12} =$ 200 pounds per square inch tension or compression. The maximum intensity of stress in the column will be $200 + 100 = 300$ pounds per square inch and will be compression. This stress will occur anywhere along the right side of the column. On the left side the stress will be tension and will be $200 - 100 = 100$ pounds per square inch.

It will be noted that the intensity of stress in the cross-section of a column loaded parallel to its axis depends very largely upon the position of the load. If the load is at the center, the intensity of stress is uniform, but if the load is not at the center, the intensity of stress varies uniformly.

AA' (Fig. 100) is the cross-section of a column loaded with a load parallel to the axis of the column but not at the center. Compression stress is plotted above AA' and tension below. The load at the center will cause a compression stress of uniform intensity Aa. The couple will cause a stress that will vary uniformly from the center of gravity of the column section AA'. This stress is shown by line bc. Combine the direct stress of uniform intensity Aa with the bending stress of uniformly varying intensity from O which is at the center of gravity of the cross-section. The intensity of the combined stress is shown by the line df. The intensity of compression at A' is $A'a + A'b = A'd$ and the intensity of the tension at A is $Ac - Aa = Af$. The intensity of stress at A may be compression, tension, or zero depending

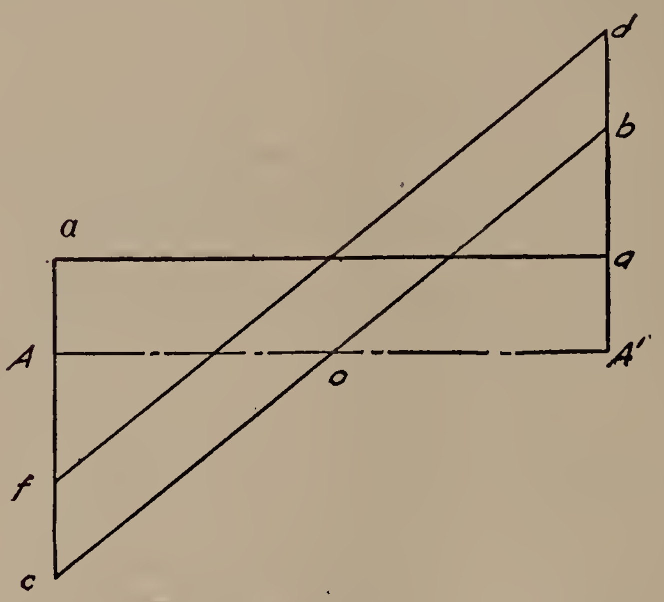


FIG. 100.

upon the relative magnitude of the bending and direct stresses. A general formula for this case is

$$f = \frac{P}{A} \pm \frac{Px_0 \times y}{I}$$

Where

P is the load on the column.

A is the area of the cross-section of column.

x_0 is the distance from the center of gravity of the cross-section of the column to the line of action of P.

y and I are the same as for any beam both being referred to the neutral axis of the cross-section of the column for bending. The neutral axis passes through the center of gravity of the cross-section and is perpendicular to the plane of the forces.

x_0 is known as the eccentricity of the load.

In the design of columns great care must be exercised to determine the greatest eccentricity of loading. It often happens that a comparatively light load with considerable eccentricity will cause a much greater stress than a heavier load at or near the center.

Columns with load not in plane of symmetry of cross-section.

The column of Fig. 101 is loaded at *a* with a vertical load $P = 3600$ pounds. Find the maximum intensity of stress.

Apply at *b* two equal and opposite vertical forces each equal to 3600 pounds. The original downward force at *a* and the upward one at *b* will form a couple in a plane parallel to XX. This will leave a single downward force at *b*. Now apply two equal and opposite vertical forces at O each equal to 3600 pounds. The downward force at *b* and the upward force at O will form a couple in the plane YY. This will leave a single downward force at O equal to 3600 pounds. This force at O will cause a compression of uniform intensity over the section. This compression will be $\frac{3600}{6 \times 12} = 50$ pounds per square inch.

The moment of the couple in plane YY will be $3600 \times 9 =$ 32,400 inch pounds.

The bending stress caused by this couple will be

$$f = \frac{My}{I} = \frac{32{,}400 \times 3 \times 12}{12 \times 6 \times 6 \times 6} = 450 \text{ pounds per square inch}$$

This couple is in the plane YY. Therefore, the neutral axis for the bending caused by it must be along XX which is perpendicular to the plane containing the bending forces.

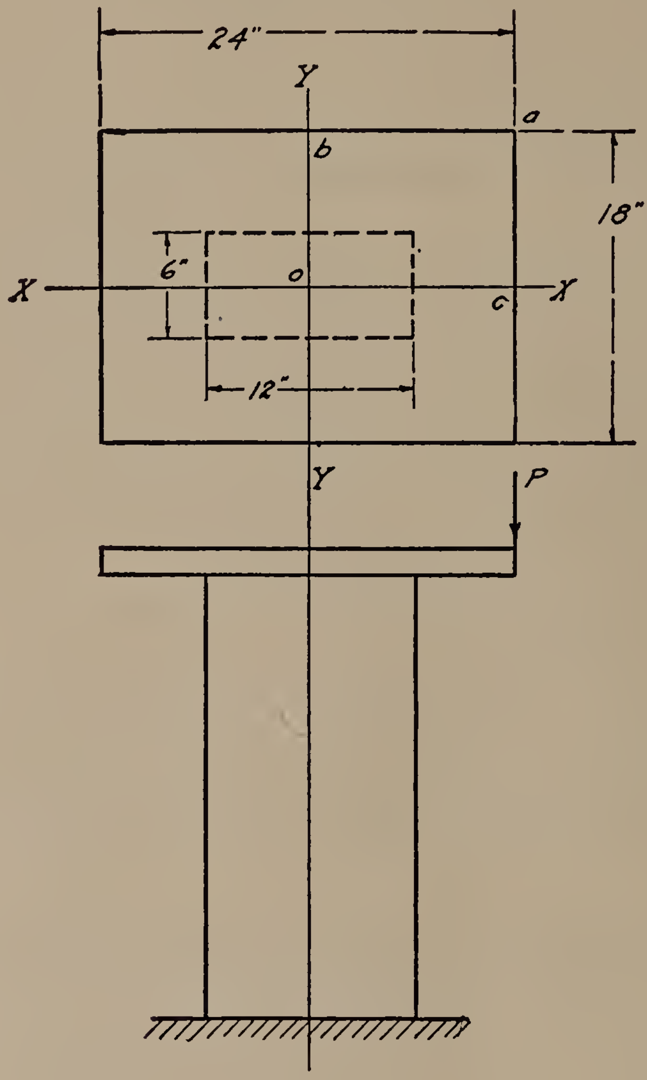


Fig. 101.

The moment of the couple in the plane parallel to XX is $3600 \times 12 = 43{,}200$ inch pounds. This couple has the same effect as one of the same moment in the plane XX.

The fiber stress caused by this couple will be:—.

$$f = \frac{My}{I} = \frac{43{,}200 \times 6 \times 12}{6 \times 12 \times 12 \times 12} = 300 \text{ pounds per square inch}$$

Fig. 102 shows the cross-section of the column.

The couple in the plane YY will cause a compression along mk of 450 pounds per square inch and a tension of the same magnitude along hg. The couple in the plane parallel to XX will cause a compression of 300 pounds per square inch along kg and a tension of the same amount along mh. The load at the center will cause a uniform compression of 50 pounds per square inch.

The intensity of stress at corner k will be $450 + 300 + 50 = 800$ pounds per square inch compression. The stress at corner g will be $450 - 300 - 50 = 100$ pounds per square inch tension.

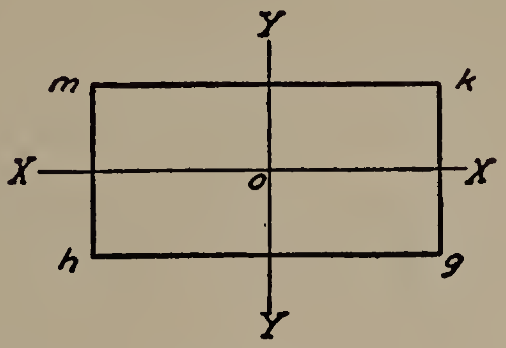


Fig. 102.

At corner h, intensity of stress will be $450 + 300 - 50 = 700$ pounds per square inch tension and at corner m the stress will be $450 - 300 + 50 = 200$ pounds per square inch compression. The intensity of stress at the center of the column will be 50 pounds per square inch compression.

Therefore maximum intensity of stress in the column is 800 pounds per square inch compression at edge k.

When the Plane Containing the Loads Does Not Divide the Cross-Section Symmetrically. Fig. 103*a–b–c* are the cross-sections of three beams each loaded with a load P which acts at right angles to the axis of the beam. In beam a, the line of action of the load passes through the center of gravity of the cross-section of the beam. The load P may be resolved into two components parallel to the axes XX and YY. If each of these components is resolved into a single force at the center of gravity and

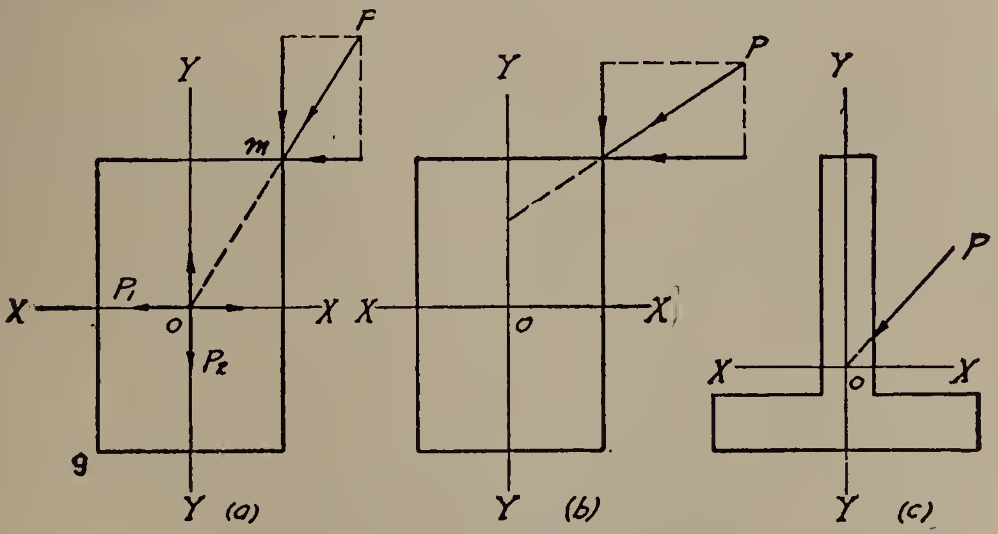


Fig. 103.

a couple, there will be two couples, the algebraic sum of the moments of which is zero and two forces acting along XX and YY both of which are axes of symmetry.

Each of these two component forces at O comes under the limitations of the common beam theory. The stress caused by each may be determined by the use of the formula $f = \frac{My}{I}$. YY is the neutral axis for P_1 and XX is the neutral axis for P_2. The corners m and g will have the sum of the two bending stresses.

In Fig. 103*b* the line of action of the force P does not pass through the center of gravity of the section, therefore the stress cannot be found as the algebraic sum of the moments of the couples will not be zero. There will be stresses due to twisting as well as bending.

In Fig. 103*c* the stress cannot be determined, although the line of action of P passes through the center of gravity of the section. There are not two axes of symmetry. Therefore, P cannot be resolved into components that will come under the limitations of the common beam theory. In order that the stresses may be determined exactly by means of the ordinary beam theory, the line of action of the forces must pass through the center of gravity of the cross-section and the cross-section must have at least two axes of symmetry.

When the beam is not straight, the beam theory does not hold without modification for curved beams. If the curvature is not very great, however, the theory will hold with slight approximation provided that the shape of the beam is such that the center axis always lies in the same plane as the one containing the loads.

The beam (Fig. 104*a*) stresses can be obtained with slight approximation if the radius of curvature is comparatively small.

The beam (Fig. 104*b*) stresses cannot be obtained.

When the Cross-section Varies the Stresses in Beams can be determined provided that the plane containing the loads always divides every cross-section symmetrically and the line passing through the center of gravity of every cross-section is approximately straight.

When the beam is not homogeneous, the determination of bending stresses becomes complex and it is not easy to obtain an exact solution. A discussion of this case is rather out of place here and should be taken up in connection with the design of reinforced concrete.

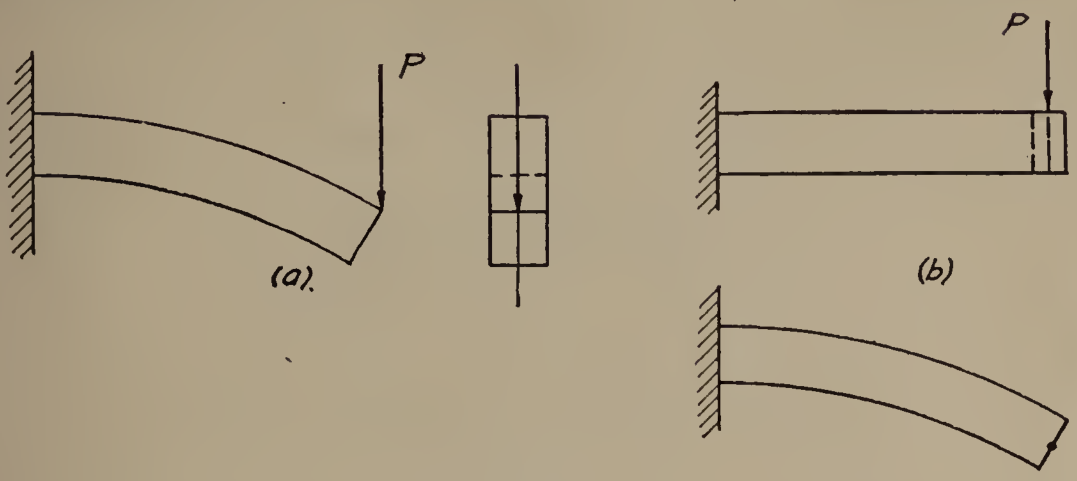


FIG. 104.

An approximate solution of a problem of this kind is given below. A wooden beam 6″ × 12″ and a 12″ I beam are used side by side as shown in Fig. 105. The combination is loaded at the center with a load of 10,000 pounds. Find the intensity of stress in each beam. Moment of inertia of I beam = 200 In.^{4}

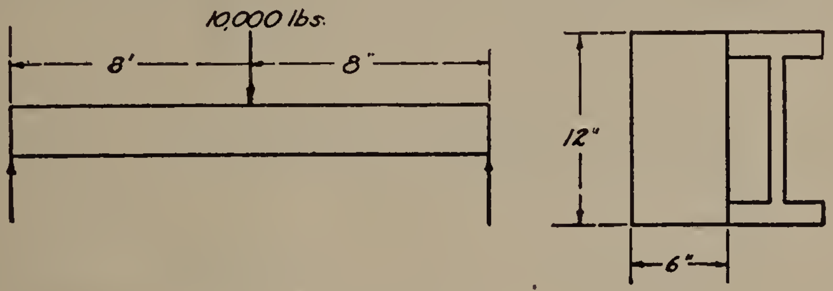


FIG. 105.

$$E_{\text{steel}} = 30{,}000{,}000 \qquad E_{\text{wood}} = 1{,}000{,}000$$

Use the fact that the deflection of the two beams must be the same.

Then
$$W_{\text{wood}} + W_{\text{steel}} = 10{,}000$$

$$\frac{1}{48}\frac{W_{\text{wood}}\, l^3}{E_{\text{wood}}\, I_{\text{wood}}} = \frac{1}{48}\frac{W_{\text{steel}}\, l^3}{E_{\text{steel}}\, I_{\text{steel}}}$$

From these two equations W_{wood} and W_{steel} can be determined. The intensities of stress in the two beams can be found from these values for the loads.

When the modulus of elasticity of the material of which the beam is made is not the same for tension as for compression the beam theory does not hold directly as the neutral axis of the stress will not be at the center of gravity of the cross-section.

25. Bearing Plates. An important problem in construction is the determination of the maximum intensity of stress upon bearing plates. The plate AB (Fig. 106) rests on the abutment C. The load P is applied at the center of gravity of the plate. The stress between the plate and C is uniformly distributed and its intensity is equal to $P \div \text{area} = Aa$.

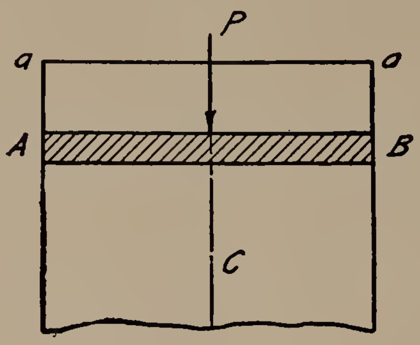


Fig. 106.

If the load P is moved from the center of the plate a distance x along an axis of symmetry as shown (Fig. 107) the intensity of stress between plate and abutment is not uniformly distributed. The assumption usually made is that the stress varies uniformly in intensity as shown by ab. The distance, x, is known as the eccentricity of the load.

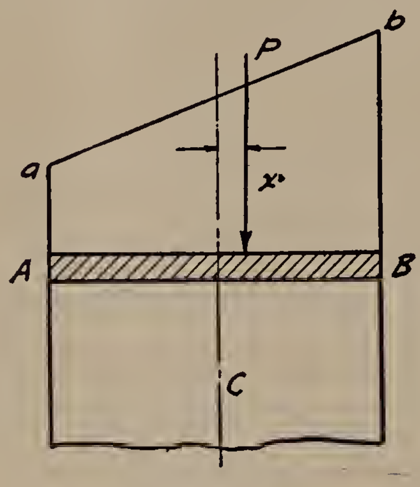


Fig. 107.

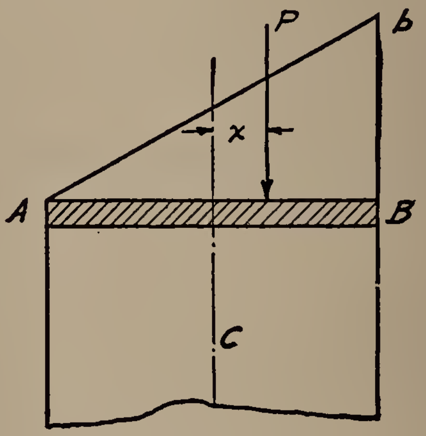


Fig. 108.

If x is made larger there will be some value of x that will bring about the condition shown in Fig. 108, where the intensity of stress between plate and abutment is zero at one side. If x is increased further there must be a reversal of the stress. This is impossible unless the plate is fastened to the abutment and the latter so designed as to be able to withstand tension. It is very important that the maximum x without stress reversal shall be determined for different shaped plates.

Problem: Rectangular plate (Fig. 109). Apply the method used for columns. Resolve force P into a force equal and parallel to it at center of gravity of plate O and a right-handed couple, the moment of which is P times the eccentricity x. The downward force at O will cause a stress of uniform intensity shown by the line mg. The couple will cause a uniformly varying stress, the intensity of which is shown by the line mn. If the intensity zg is equal and opposite to the intensity zn, the resultant intensity of stress at z will be zero and the intensity at z' will be $z'k = z'm + zg$. To find this limiting value of x when the bearing surface is rectangular

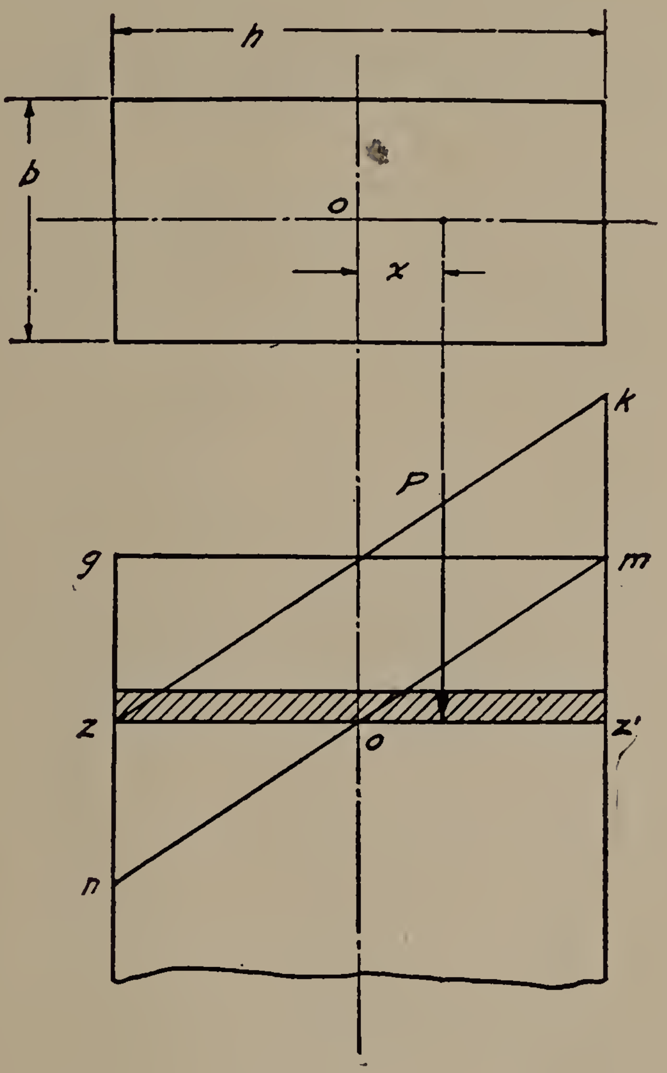


FIG. 109.

$$\frac{P}{bh} = \frac{(Px)\frac{h}{2}12}{bh^3} = \frac{Px\,6}{bh^2} \qquad \text{Therefore} \quad x = \frac{h}{6}$$

If the resultant load is at or within the middle third of the area over which the stress is distributed there will be no tendency for the plate to lift off the support or no tension in the support if the plate is fastened to it.

If the load is outside the middle third (Fig. 110) and there can be no tension the following method is useful. If there can be no tension the load must be at or within the middle third of the area over which the stress is distributed. In this case the load can be distributed over $AB = 9''$ which will bring the load at the middle third of the area AB. To find the maximum intensity of compression:

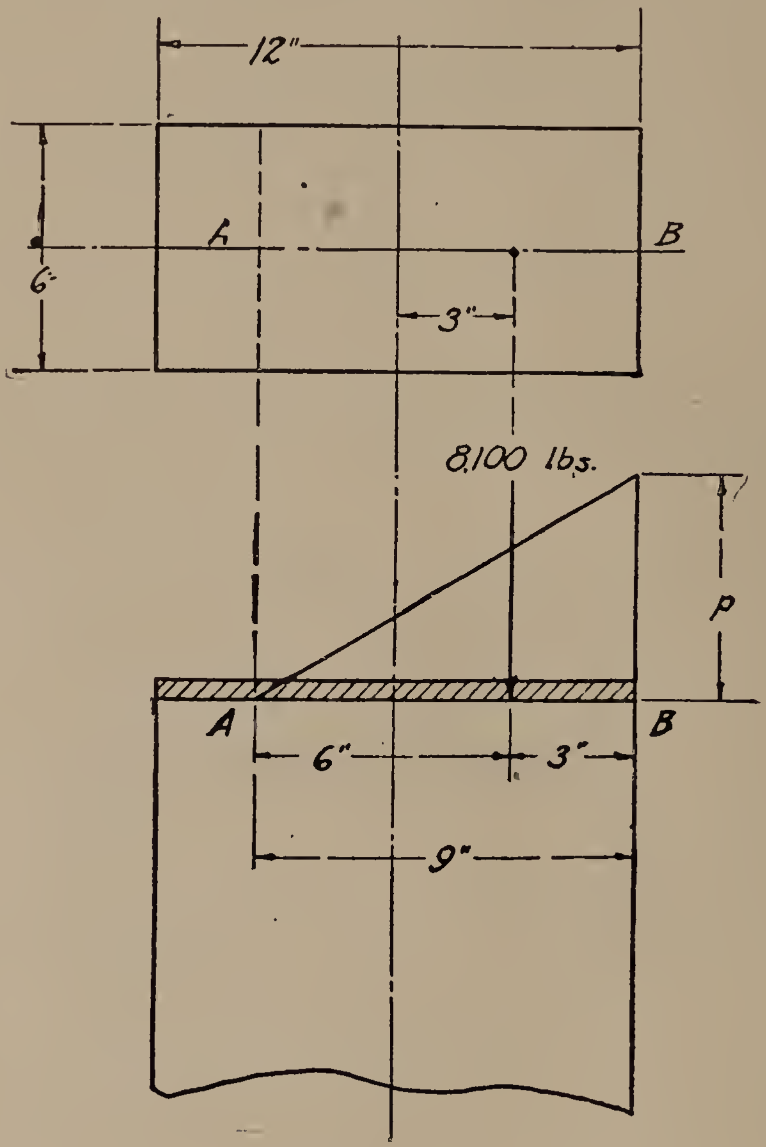


FIG. 110.

$$\frac{p \times 6 \times 9}{2} = 8100$$

$$\text{Therefore} \quad p = \frac{8100 \times 2}{6 \times 9} = 300 \text{ pounds per square inch.}$$

The limits for the application of the load for no tension for any shape of plate can be determined in the same way.

If the bearing plate does not extend over the whole width of the supporting wall or abutment, the stress in the latter can be obtained in the same manner as for columns with eccentric loading.

26. Conclusion. Any problem in construction will probably bring up conditions that will depart from the limitations set down here. The methods of analysis and solution of such problems are the function of Courses in Structural Design.

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